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Disturbance Decoupling of Discrete-time Nonlinear Systems by Static Measurement Feedback

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Abstract: This paper addresses the disturbance decoupling problem (DDP) for nonlinear systems extending the results for continuous-time systems into the discrete-time case. Sufficient conditions are given for the solvability of the problem. The notion of the rank of a one-form is used to find the static measurement feedback, that solves the DDP whenever possible. Moreover, necessary and sufficient conditions are given for single-input single-output systems as well as for multi-input single-output systems under the additional assumption.

Keywords: Nonlinear systems, discrete-time systems, disturbance decoupling, static measurement feedback.

1. INTRODUCTION

The disturbance decoupling problem (DDP) for discrete-time nonlinear control system by state feedback has been addressed in many papers; see Aranda-Bricaire and Kotta (2004 2001); Fliegner and Nijmeijer (1994); Grizzle (1985); Kotta and Nijmeijer (1991); Monaco and Normand-Cyrot (1984). Most papers extend the known results for continuous-time systems (see for example Nijmeijer and van der Schaft (1990); Conte et al. (2007); Isidori (1995)) into the discrete-time domain and in all these papers the control system is described by smooth or analytic difference equations. However, there are no papers that address the DDP for discrete-time nonlinear control systems using the output feedback except that of by Shumsky and Zhirabok (2010) (see also Kotta and Mullari (2010)) and Kotta et al. (2011). In Shumsky and Zhirabok (2010) the controlled output is a vector function of the measured output, having possibly less components than the measured output itself. Therefore, the above solution may be considered only as a partial solution. The paper by Kotta et al. (2011) provides a full algorithmic solution for the problem using the dynamic feedback. In both papers the novel algebraic approach, called the algebra of functions, is applied, see Zhirabok and Shumsky (2008).

As for continuous-time nonlinear control systems there exist also only a few papers addressing the problem, see Pothin et al. (2002); Isidori et al. (1981); Xia and Moog (1999); Andiarti and Moog (1996). The paper by Pothin et al. (2002) studies the problem using a static measurement feedback, and in Isidori et al. (1981) the feedback considered is restricted to pure dynamic measurement feedback, whereas the other two papers focus on the dynamic measurement feedback.

The goal of this paper is to extend the results of Pothin et al. (2002) for discrete-time nonlinear control systems.

2. PROBLEM STATEMENT

Consider a discrete-time nonlinear control system

$$\begin{aligned} x(t+1) &= f(x(t), u(t), w(t)) \\ y(t) &= h(x(t)) \\ z(t) &= k(x(t)), \end{aligned} \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$, the control input $u(t) \in \mathbb{R}^m$, the disturbance input $w(t) \in \mathbb{R}^\nu$, the output to be controlled $y(t) \in \mathbb{R}$ and the measured output $z(t) \in \mathbb{R}^\mu$. Assume that f, h and k are meromorphic functions of their arguments.

Let $\mathcal{K}^*$ denote the inversive field of meromorphic functions in variables $x(t)$, $u(t)$, $w(t)$ and a finite number of their (independent) forward- and backward shifts. Note that not all the variables are independent because of the relationships defined by (1) and in the computations the dependent variables have to be expressed via the independent ones. For example, $x(t+1)$ has to be replaced by $f(x(t), u(t), w(t))$. See Aranda-Bricaire et al. (1996) for the details how to construct $\mathcal{K}^*$.

Define the vector spaces $\mathcal{X} = \operatorname{span}_{\mathcal{K}^*}\{dx(t)\}$, $\mathcal{Z} = \operatorname{span}_{\mathcal{K}^*}\{dz(t)\}$, $\mathcal{U} = \operatorname{span}_{\mathcal{K}^*}\{du(t+k), k \geq 0\}$, $\mathcal{W} = \operatorname{span}_{\mathcal{K}^*}\{dw(t+k), k \geq 0\}$ and $\mathcal{E} = \mathcal{X} + \mathcal{U} + \mathcal{W}$.

Definition 1. (Aranda-Bricaire et al. (1996)) The relative degree r of the output $y(t)$ is defined by

$$r := \min\{k \in \mathbb{N} | dy(t+k) \notin \mathcal{X}\}.$$

If such an integer does not exist, then define $r := \infty$.

The static measurement feedback of the form $u(t) = F(z(t), v(t))$ is called regular if F is invertible with respect to $v(t)$, i.e. if there exists an inverse function $\alpha := F^{-1}$ such that $v(t) = \alpha(z(t), u(t))$.

Problem Statement. Given a nonlinear system of the form (1), the goal is to find, if possible, a regular static measurement feedback of the form

$$u(t) = \alpha^{-1}(z(t), v(t)),$$

such that controlled output $y(t)$ of the closed loop system satisfies the following conditions:

(i) $\mathrm{d}y(t+k) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}v(t), \dots, \mathrm{d}v(t+k-r)\}, \forall k \geq r$
(ii) $\mathrm{d}y(t+r) \notin \mathcal{X}$.

Condition (i) represents the independence of the output of the closed-loop system from the disturbance whereas the condition (ii) represents the output controllability of the closed-loop system.

Analogously to the continuous-time case, see Pothin et al. (2002), define the subspace $\Omega \subset \mathcal{X}$ by

$$\begin{aligned}\Omega := \{&\omega(t) \in \mathcal{X} | \forall k \in \mathbb{N} : \omega(t+k) \\ &\in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}y(t+r), \dots, \mathrm{d}y(t+r+k-1)\}\}.\end{aligned}$$

Lemma 1. The subspace Ω may be computed as the limit of the following algorithm:

$$\begin{aligned}\Omega^0 &= \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t)\} \\ \Omega^{k+1} &= \{\omega(t) \in \Omega^k | \omega(t+1) \in \Omega^k \\ &\quad + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r)\}\}, \qquad k \geq 0.\end{aligned} \tag{2}$$

Proof: We show below, that sequence Ω^k converges and in the limit we get Ω. Consider a subspace Ω^k. By (2), $\Omega^{k+1} \subset \Omega^k$ or $\Omega^{k+1} = \Omega^k$. Since the subspace Ω^k is finite-dimensional vector space, at certain step k^*+1, $\Omega^{k^*} = \Omega^{k^*+1}$. Thus the sequence (2) converges and the limit is Ω^{k^*}. We show now that $\Omega = \Omega^{k^*}$. Suppose $\omega(t) \in \Omega^{k^*}$. Then, by (2)

$$\omega(t+1) \in \Omega^{k^*-1} + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r)\}$$

and so $\omega(t+1) = \tilde{\omega}(t) + \xi \mathrm{d}y(t+r)$ for some $\tilde{\omega}(t) \in \Omega^{k^*-1}$ and function $\xi \in \mathcal{K}^*$. Since $\tilde{\omega}(t) \in \Omega^{k^*-1}$, by (2)

$$\tilde{\omega}(t+1) \in \Omega^{k^*-2} + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r)\}$$

and so forward shift of $\omega(t+1)$ is

$$\omega(t+2) \in \Omega^{k^*-2} + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r), \mathrm{d}y(t+r+1)\}.$$

Continuing in the same way, we get

$$\omega(t+k^*) \in \Omega^0 + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r), \dots, \mathrm{d}y(t+r+k^*-1)\},$$

which means that $\omega(t) \in \Omega$. We showed that if $\omega(t) \in \Omega^{k^*}$, then $\omega(t) \in \Omega$, i.e. $\Omega^{k^*} \subset \Omega$.

Now suppose that $\omega(t) \in \Omega$. Then by definition of Ω,

$$\omega(t+k^*) \in \mathcal{X} + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r), \dots, \mathrm{d}y(t+r+k^*-1)\}.$$

Because $\Omega^0 = \mathcal{X}$,

$$\omega(t+k^*) = \tilde{\omega}(t) + \xi_1 \mathrm{d}y(t+r) + \ldots + \xi_{k^*} \mathrm{d}y(t+r+k^*-1),$$

where $\tilde{\omega}(t) \in \Omega^0$ and $\xi_1, \dots, \xi_{k^*} \in \mathcal{K}^*$. Backward shift $\tilde{\omega}(t-1) \in \Omega^1$, because $\tilde{\omega}(t-1) \in \Omega^0$ and $\tilde{\omega}(t) \in \Omega^0 + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r)\}$. Note that $\mathrm{d}y(t+r-1) \in \Omega^{k^*}$, because $\mathrm{d}y(t+r) \in \Omega^l + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r)\}$ for every $l \geq 0$. Thus backward shift of $\omega(t+k^*)$ is

$$\omega(t+k^*-1) \in \Omega^1 + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r), \dots, \mathrm{d}y(t+k^*-2)\}.$$

Continuing in the same way, we get

$$\omega(t+1) \in \Omega^{k^*-1} + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+r)\}.$$

Thus $\omega(t) \in \Omega^{k^*}$ and we are shown that $\Omega \subset \Omega^{k^*}$. Above we showed that $\Omega^{k^*} \subset \Omega$, so $\Omega = \Omega^{k^*}$. □

We will show next how Ω changes under the regular static measurement feedback $u(t) = \alpha(z(t), v(t))$. Denote by $\overline{\mathcal{K}^*}$ the field of meromorphic functions in variables $x(t)$, $v(t)$, $w(t)$ and a finite number of their independent forward- and backward shifts and define the vector spaces $\overline{\mathcal{X}} = \mathrm{span}_{\overline{\mathcal{K}^*}}\{\mathrm{d}x(t)\}$, $\overline{\mathcal{U}} = \mathrm{span}_{\overline{\mathcal{K}^*}}\{\mathrm{d}u(t+k), k \geq 0\}$, $\overline{\mathcal{W}} = \mathrm{span}_{\overline{\mathcal{K}^*}}\{\mathrm{d}w(t+k), k \geq 0\}$, $\overline{\mathcal{E}} = \overline{\mathcal{X}} + \overline{\mathcal{U}} + \overline{\mathcal{W}}$. Analogously to Xia and Moog (1999) one can prove that there exists an isomorphism $\Phi : \mathcal{E} \to \overline{\mathcal{E}}$ such that if Ω_{cl} is the subspace for the closed loop system, then $\Omega_{cl} = \Phi(\Omega)$.

Let $\omega(t) \in \Theta$ be a one-form. In general, $\omega(t)$ is a linear combination of all n basis elements of Θ, i.e. $\{\theta_1, \dots, \theta_n\}$. However, it is often possible to find a linearly independent subset of the set $\{\theta_1, \dots, \theta_n\}$ with less than n elements in terms of which $\omega(t)$ can be expressed.

Definition 2. (Choquet-Bruhat et al. (1996)) Let γ be the minimal number of linearly independent one-forms necessary to express a one-form $\omega(t)$. Then $\omega(t)$ is said to be of rank γ.

Note that $1 \leq \gamma \leq n$. For example, if the rank γ of a one-form $\omega(t)$ is 1, then $\omega(t) = \xi \mathrm{d}\alpha$ and thus $\omega(t) \wedge \mathrm{d}\omega(t) = 0$. In the general case, if the rank γ is k, then $\omega(t) \wedge (\mathrm{d}\omega(t))^{(k)} = 0$, where $(\mathrm{d}\omega(t))^{(k)} = \mathrm{d}\omega(t) \wedge \ldots \wedge \mathrm{d}\omega(t)$ is k-fold wedge product.

We prove the following lemma for MISO systems, providing the alternative formulation of the disturbance decoupling.

Lemma 2. Under the assumption that the relative degree r of the output $y(t)$ is finite, the system (1) is disturbance decoupled iff

$$\mathrm{d}y(t+r) \in \Omega + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}u(t)\}. \tag{3}$$

Proof: *Necessity.* Assume, that system (1) is disturbance decoupled, i.e.

$$\mathrm{d}y(t+k) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t), \dots, \mathrm{d}u(t+k-r)\} \tag{4}$$

for $k \geq r$ and

$$\mathrm{d}y(t+r) \notin \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t)\}. \tag{5}$$

In particular, $\mathrm{d}y(t+r) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t)\}$. Rewrite the latter as

$$\mathrm{d}y(t+r) \in \mathcal{X} + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}u(t)\}. \tag{6}$$

Thus there exists a one-form $\omega_0(t) \in \mathcal{X}$ and a function $\xi \in \mathcal{K}^*$ such that $\mathrm{d}y(t+r) = \omega_0(t) + \xi \mathrm{d}u(t)$. We are going to show, that $\omega_0(t) \in \Omega$. Assume contrarily, that $\omega_0(t) \notin \Omega$. The forward shift of $\mathrm{d}y(t+r) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t)\}$ is

$$\mathrm{d}y(t+r+1) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}w(t), \mathrm{d}u(t), \mathrm{d}u(t+1)\},$$

which yields a contradiction with (4). Thus, $\omega_0 \in \Omega$ and we can rewrite (6) as $\mathrm{d}y(t+r) \in \Omega + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}u(t)\}$.

Sufficiency. Assume that for system (1) the condition (3) is fulfilled. We must show that system (1) satisfies conditions (4) and (5). Because r is the relative degree of $y(t)$, (5) is satisfied. Because of (3),

$$\mathrm{d}y(t+r) = \omega_0(t) + \xi \mathrm{d}u(t),$$

where $\omega_0(t) \in \Omega$ and $\xi \in \mathcal{K}^*$. Since $\omega_0(t) \in \Omega$

$$\omega_0(t+l) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}y(t+r), \dots, \mathrm{d}y(t+r+l-1)\}$$

for all $l \geq 0$. Thus

$$\mathrm{d}y(t+r+l) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}y(t+r), \ldots \\ \ldots, \mathrm{d}y(t+r+l-1), \mathrm{d}u(t+l)\}$$

for all $l \geq 0$. Hence

$\mathrm{d}y(t+r+l-1) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}y(t+r), \ldots, \mathrm{d}y(t+r+l-2)\}$ and

$$\mathrm{d}y(t+r+l) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}y(t+r), \ldots \\ \ldots, \mathrm{d}y(t+r+l-2), \mathrm{d}u(t+l-1), \mathrm{d}u(t+l)\}.$$

Continuing the same way, we get

$$\mathrm{d}y(t+r+l) \in \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t), \ldots, \mathrm{d}u(t+l)\}.$$

Changing l by $l = k - r$, we get (4) and thus sufficiency is fulfilled. □

We are going to use the subspace Ω and the concept of the rank of a one-form to give a sufficient condition for the disturbance decoupling problem.

3. MAIN RESULTS

Theorem 3. The disturbance decoupling problem for system (1) is solvable by static measurement feedback if:

(i) $\mathrm{d}y(t+r) \in \Omega + \mathcal{Z} + \mathcal{U}$,
(ii) there exists a one-form $\omega(t) \in \mathcal{Z}+\mathcal{U}$ such that $\mathrm{d}y(t+r) - \omega(t) \in \Omega$ and rank $\omega(t) = \gamma \leq m$,
(iii) for any basis $\{\mathrm{d}\alpha_1(z(t), u(t)), \ldots, \mathrm{d}\alpha_\gamma(z(t), u(t))\}$ of $\omega(t)$,

$$\mathrm{rank}\Big[\frac{\partial \alpha(z(t), u(t))}{\partial u(t)}\Big] = \gamma, \tag{7}$$

where $\alpha := [\alpha_1, \ldots, \alpha_\gamma]^T$.

Proof: Assume that condition (i) is fulfilled. Under the condition (ii) there exists a one-form $\omega(t)$ such that $\mathrm{d}y(t+r) - \omega(t) \in \Omega$ where

$$\omega(t) = \beta_1 \mathrm{d}\alpha_1(z(t), u(t)) + \ldots + \beta_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t)).$$

When condition (iii) is satisfied, then γ one-forms $\mathrm{d}\alpha_i(z(t), u(t))$, $i = 1, \ldots, \gamma$, are independent with respect to the variable $u(t)$. Define for $i = 1, \ldots, \gamma$

$$v_i(t) = \alpha_i(z(t), u(t)). \tag{8}$$

If $\gamma < m$, then by renumbering the inputs $u(t)$, if necessary, complete (8) with

$$v_i(t) = u_i(t), \qquad i = \gamma+1, \ldots, m \tag{9}$$

to get an invertible map. Define a static measurement feedback $u(t) = \alpha(z(t), u(t))$ as the solution of (8) and (9). Note that this yields

$$\mathrm{d}y(t+r) \in \Omega \oplus \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}v(t)\}$$

and thus by Lemma 2, system (1) is disturbance decoupled. □

In case of SISO systems when $m = 1$, (7) and (ii) of Theorem 3 yield

$$\mathrm{rank}\Big[\frac{\partial \alpha(z(t), u(t))}{\partial u(t)}\Big] = \gamma = 1.$$

Thus, condition (iii) of Theorem 3 is satisfied if and only if $\gamma = 1$. For SISO systems one can conclude from Theorem 3 a necessary and sufficient condition.

Corollary 4. For SISO nonlinear control systems the DDP is solvable by a regular static measurement feedback iff:

(i) $\mathrm{d}y(t+r) \in \Omega + \mathcal{Z} + \mathcal{U}$
(ii) There exists a one-form $\omega(t) \in \mathcal{Z} + \mathcal{U}$ such that $\mathrm{d}y(t+r) - \omega(t) \in \Omega$ and rank $\omega(t) = 1$.

Proof: *Necessity.* Assume that system (1) is decoupled by the regular static measurement feedback

$$u(t) = \alpha(z(t), v(t)), \qquad v(t) = \alpha^{-1}(z(t), u(t)). \tag{10}$$

Then by Lemma 2

$$\mathrm{d}y(t+r) \in \Omega + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}v(t)\}. \tag{11}$$

Combining (11) with (10) implies condition (i). Since $\omega(t) = \xi \mathrm{d}(F^{-1}(z(t), u(t)))$, $\omega(t) \wedge \mathrm{d}\omega(t) = 0$ and rank $\omega(t) = 1$. Thus condition (ii) is also fulfilled.

Sufficiency. Assume that (i) holds. Then

$$\mathrm{d}y(t+r) \in \Omega \oplus \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}z(t), \mathrm{d}u(t)\}.$$

Since by (ii) the rank of the one-form $\omega(t)$ is 1, define $\omega(t) := \lambda \mathrm{d}v(t)$ and so

$$\mathrm{d}y(t+r) \in \Omega \oplus \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}v(t)\}$$

meaning that the system is decoupled. □

In general there is no necessary and sufficient condition for MISO systems, but under an additional assumptions $\Omega \cap \mathcal{Z} = \emptyset$ and $\mathrm{d}y(t+r) \in \Omega \oplus \mathcal{Z} + \mathcal{U}$ one can find a necessary and sufficient condition for MISO systems.

Theorem 5. Assume that $\Omega \cap \mathcal{Z} = \emptyset$ and $\mathrm{d}y(t+r) \in \Omega \oplus \mathcal{Z}+\mathcal{U}$. The DDP is solvable by regular static measurement feedback iff

(i) There exists a one-form $\omega(t) \in \mathcal{Z} + \mathcal{U}$ such that $\mathrm{d}y(t+r) - \omega(t) \in \Omega$ and $\gamma := \mathrm{rank}\ \omega(t) \leq m$.
(ii) For any basis $\{\mathrm{d}\alpha_1(z(t), u(t)), \ldots, \mathrm{d}\alpha_\gamma(z(t), u(t))\}$ of $\omega(t)$,

$$\mathrm{rank}\Big[\frac{\partial \alpha(z(t), u(t))}{\partial u(t)}\Big] = \gamma.$$

Proof: *Necessity.* Assume that system (1) is disturbance decoupled by the regular static measurement feedback $v(t) = \alpha(z(t), u(t))$. By Lemma 2, $\mathrm{d}y_{cl}(t+r) \in \Omega_{cl} + \mathcal{V}$, where $\mathcal{V} = \mathrm{span}_{\overline{\mathcal{K}^*}}\{\mathrm{d}v_1(t), \ldots, \mathrm{d}v_m(t)\}$ and $y_{cl}(t)$ is the output of the closed-loop system. Because of isomorphism $\Phi : \mathcal{E} \to \overline{\mathcal{E}}$ described above and feedback $\alpha(z(t), u(t))$, one can write

$$\mathrm{d}y(t+r) \in \Omega + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}\alpha(z(t), u(t))\}.$$

Thus, there exist a one-form $\tilde{\omega}(t) \in \Omega$ and $\xi \in \mathcal{K}^*$ such that

$$\mathrm{d}y(t+r) = \tilde{\omega}(t) + \xi \mathrm{d}\alpha(z(t), u(t)).$$

Assumption $\mathrm{d}y(t+r) \in \Omega \oplus \mathcal{Z}+\mathcal{U}$ implies that $\tilde{\omega}(t) \in \Omega + \mathcal{Z}$. Rewrite $\tilde{\omega}(t) = \tilde{\omega}_0(t) + \tilde{\omega}_z(t)$ for some $\tilde{\omega}_0(t) \in \Omega$ and $\tilde{\omega}_z(t) \in \mathcal{Z}$. As in the proof of Lemma 2, one can show that $\tilde{\omega}_z(t) \in \Omega$. Due to the assumption $\Omega \cap \mathcal{Z} = 0$, we have $\tilde{\omega}_z(t) = 0$. Then define $\omega(t) = \xi \mathrm{d}\alpha(z(t), u(t))$ and the necessity of condition (i) is fulfilled.

Because the rank of a one-form $\omega(t)$ is γ,

$$\omega(t) = \beta_1 \mathrm{d}\alpha_1(z(t), u(t)) + \ldots + \beta_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t))$$

where $\beta_i \in \mathcal{K}^*$, $i = 1, \ldots, \gamma$. Suppose, contrarily to the claim of Theorem 5 that (ii) is not fulfilled. Then there exist a one-form

$$\xi_1 \mathrm{d}\alpha_1(z(t), u(t)) + \ldots + \xi_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t)) \in \mathcal{Z}.$$

Assume without loss of generality that $\xi_1 \neq 0$, then $\omega(t)$ can be decomposed into

$$\omega(t) = \tilde{\omega}_z(t) + \eta_2 \mathrm{d}\alpha_2(z(t), u(t)) + \ldots + \eta_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t))$$

in which

$$\tilde{\omega}_z(t) = \frac{\beta_1}{\xi_1}(\xi_1 \mathrm{d}\alpha_1(z(t), u(t)) + \ldots + \xi_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t))) \in \mathcal{Z}$$

and

$$\eta_i = \beta_i - \frac{\beta_1}{\xi_1}\xi_i,$$

for $i = 2, \ldots, \gamma$. As shown before, if $\tilde{\omega}_z(t) \in \mathcal{Z}$ then necessarily $\tilde{\omega}_z(t) \in \Omega$ and since $\Omega \cap \mathcal{Z} = 0$, this yields a contradiction. Thus condition (ii) has to be fulfilled.

Sufficiency. Because all of the conditions of Theorem 3 are satisfied, then sufficiency is fulfilled. □

4. EXAMPLES

The first example illustrates Theorem 3.

Example 1. Consider the system

$$\begin{aligned}
x_1(t+1) &= x_2(t) + x_3(t)u_1(t)x_4(t) + u_2(t)x_4(t)\\
x_2(t+1) &= x_2(t) + x_3(t)u_1(t)x_4(t) + u_2(t)x_4(t) + x_3^2(t)\\
x_3(t+1) &= \cos x_1(t) \qquad (12)\\
x_4(t+1) &= w(t)\\
y(t) &= x_1(t)\\
z(t) &= x_4(t).
\end{aligned}$$

Note that the relative degree of the output $y(t)$ is 1, because

$$\begin{aligned}
\mathrm{d}y(t+1) = {} & \mathrm{d}x_2(t) + u_1(t)x_3(t)\mathrm{d}x_4(t) + x_3(t)x_4(t)\mathrm{d}u_1(t)\\
& + u_1(t)x_4(t)\mathrm{d}x_3(t) + u_2(t)\mathrm{d}x_4(t) + x_4(t)\mathrm{d}u_2(t).
\end{aligned}$$

Next we find the vector space Ω using the algorithm, defined by (2). First,

$$\Omega^0 = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x_1(t), \mathrm{d}x_2(t), \mathrm{d}x_3(t), \mathrm{d}x_4(t)\}.$$

Because $\mathrm{d}y(t+1) = \mathrm{d}x_1(t+1)$,

$$\begin{aligned}
\mathrm{d}x_1(t+1) &= \mathrm{d}y(t+1) \in \Omega^0 + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+1)\}\\
\mathrm{d}x_2(t+1) &= \mathrm{d}y(t+1) + 2x_3(t)\mathrm{d}x_3(t)\\
&\in \Omega^0 + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+1)\}\\
\mathrm{d}x_3(t+1) &= -\sin x_1(t)\mathrm{d}x_1(t) \in \Omega^0 + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+1)\}\\
\mathrm{d}x_4(t+1) &= \mathrm{d}w(t) \notin \Omega^0 + \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y(t+1)\}.
\end{aligned}$$

Thus, $\Omega^1 = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x_1(t), \mathrm{d}x_2(t), \mathrm{d}x_3(t)\}$. In the next step we get $\Omega^1 = \Omega^2 = \Omega$. Since $\mathrm{d}z(t) = \mathrm{d}x_4(t)$, the condition (i) of Theorem 3 is satisfied, i.e. $\mathrm{d}y(t+1) \in \Omega + \mathcal{Z} + \mathcal{U}$. Next step is to choose $\omega(t)$ such that $\omega(t) \in \mathcal{Z} + \mathcal{U}$ and $\mathrm{d}y(t+1) - \omega(t) \in \Omega$. One can take $\omega(t) := u_2(t)\mathrm{d}x_4(t) + x_4(t)\mathrm{d}u_2(t) + u_1(t)x_3(t)\mathrm{d}x_4(t) + x_3(t)x_4(t)\mathrm{d}u_1(t)$ which can be rewritten as

$$\omega(t) = \mathrm{d}(u_2(t)z(t)) + x_3(t)\mathrm{d}(u_1(t)z(t)).$$

From above, the rank of $\omega(t)$ is $2 = m$. Thus condition (ii) of Theorem 3 is satisfied. Condition (iii) is easily verified and the disturbance decoupling feedback may be found as the solution of the system of equations $v_1(t) = u_2(t)z(t)$ and $v_2(t) = u_1(t)z(t)$ with respect to $u_1(t)$ and $u_2(t)$.

In the second example the rank of a one-form $\omega(t)$ is strictly less than the number of inputs, $\gamma < m$.

Example 2. Consider the system

$$\begin{aligned}
x_1(t+1) &= x_2(t) + x_4(t)u_1(t)u_2(t)\\
x_2(t+1) &= x_2(t) + x_4(t)u_1(t)u_2(t) + x_3^2(t)\\
x_3(t+1) &= \cos x_1(t) \qquad (13)\\
x_4(t+1) &= w(t)\\
y(t) &= x_1(t)\\
z(t) &= x_4(t).
\end{aligned}$$

The relative degree of output $y(t)$ is 1 and

$$\begin{aligned}
\mathrm{d}y(t+1) = {} & \mathrm{d}x_2(t) + u_1(t)u_2(t)\mathrm{d}x_4(t)\\
& + u_2(t)x_4(t)\mathrm{d}u_1(t) + u_1(t)x_4(t)\mathrm{d}u_2(t).
\end{aligned}$$

Like in Example 1, one can find the subspace $\Omega = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x_1(t), \mathrm{d}x_2(t), \mathrm{d}x_3(t)\}$ and thus the condition (i) of Theorem 3 is satisfied. Since now one can choose $\omega(t)$ as $\omega(t) = \mathrm{d}(u_1(t)u_2(t)z(t))$, $\gamma :=\mathrm{rank}\ \omega(t) = 1$ and condition (ii) of Theorem 3 is fulfilled. Note that (iii) is satisfied and the regular static measurement feedback can be found from $v_1(t) = u_1(t)u_2(t)z(t)$ and $v_2(t) = u_2(t)$.

The following example shows that for the MISO case the condition (i) of Theorem 3 is not necessary.

Example 3. Consider the system

$$\begin{aligned}
x_1(t+1) &= x_2(t) + u_1(t)x_3(t)x_4(t) + u_2(t)x_4(t)\\
x_2(t+1) &= x_2(t) + u_1(t)x_3(t)x_4(t) + u_2(t)x_4(t) + x_3^2(t)\\
x_3(t+1) &= u_1(t)x_4(t) \qquad (14)\\
x_4(t+1) &= w(t)\\
y(t) &= x_1(t)\\
z(t) &= x_4(t)
\end{aligned}$$

Condition (i) of Theorem 3 is not satisfied, because $\Omega = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x_1(t)\}$, but

$$\begin{aligned}
\mathrm{d}y(t+1) = {} & \mathrm{d}x_2(t) + u_1(t)x_4(t)\mathrm{d}x_3(t) + u_1(t)x_3(t)\mathrm{d}x_4(t)\\
& + x_3(t)x_4(t)\mathrm{d}u_1(t) + u_2(t)\mathrm{d}x_4(t) + x_4(t)\mathrm{d}u_2(t).
\end{aligned}$$

Still, one can choose $\omega(t) = x_3(t)\mathrm{d}(u_1(t)z(t)) + \mathrm{d}(u_2(t)z(t))$ and find the static measurement feedback from $v_1(t) = u_1(t)z(t)$ and $v_2(t) = u_2(t)z(t)$, which solves the DDP.

Example 4. Consider the system

$$\begin{aligned}
x_1(t+1) &= e^{x_2(t)x_3^2(t)}\\
x_2(t+1) &= \cos x_2(t)\\
x_3(t+1) &= u_1(t)\sin x_4(t) \qquad (15)\\
x_4(t+1) &= u_2(t)w(t)\\
y(t) &= x_1(t)\\
z(t) &= x_4(t).
\end{aligned}$$

Note that the relative degree of the output $y(t)$ is 2. Sequence (2) for this system converges and the subspace $\Omega = \Omega^2 = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x_2(t)\}$. Because

$$\mathrm{d}y(t+2) = \mathrm{d}(e^{u_1^2(t)\cos x_2(t)\sin^2 x_4(t)}),$$

the first condition of Theorem 3 is satisfied. One can choose $\omega(t)$ to be

$$\omega(t) = 2e^{u_1^2(t)\cos x_2(t)\sin^2 z(t)}(\cos x_2(t)\sin^2 z(t)u_1(t)\mathrm{d}u_1(t) + \cos x_2(t)\cos z(t)\sin z(t)u_1^2(t)\mathrm{d}z(t)).$$

Because $\omega(t) \wedge \mathrm{d}\omega(t) = 0$, the rank of $\omega(t)$ is 1 and thus the second condition is satisfied. Next, one can find $\alpha(z(t), u(t)) = \ln(u_1(t)\sin z(t))$; so rank$\left[\frac{\partial\alpha(z(t),u(t))}{\partial u(t)}\right] = 1$ and condition (iii) is also satisfied. The feedback that solves the DDP is

$$\begin{aligned} u_1(t) &= e^{v_1(t)}\csc z(t) \\ u_2(t) &= v_2(t). \end{aligned}$$

and the closed-loop system

$$\begin{aligned} x_1(t+1) &= e^{x_2(t)x_3(t)^2} \\ x_2(t+1) &= \cos(x_2(t)) \\ x_3(t+1) &= e^{v_1(t)} \\ x_4(t+1) &= v_2(t)w(t) \\ y(t) &= x_1(t) \\ z(t) &= x_4(t) \end{aligned}$$

is disturbance decoupled.

Example 5. Consider the system

$$\begin{aligned} x_1(t+1) &= x_4(t)w(t)\ln(x_2(t)u_2(t)) \\ x_2(t+1) &= x_1(t)x_2(t) \\ x_3(t+1) &= e^{u_1(t)x_4(t)} \\ x_4(t+1) &= u_2(t)w(t) \\ y(t) &= x_1(t) \\ z(t) &= x_4(t). \end{aligned} \tag{16}$$

The relative degree of output $y(t)$ is 1 and the subspace $\Omega = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x_1(t), \mathrm{d}x_2(t)\}$. Since

$$\begin{aligned} \mathrm{d}y(t+1) = {} & \frac{w(t)z(t)}{x_2(t)}\mathrm{d}x_2(t) + \frac{w(t)z(t)}{u_2(t)}\mathrm{d}u_2(t) \\ & + w(t)\ln(u_2(t)x_2(t))\mathrm{d}z(t) \\ & + z(t)\ln(u_2(t)x_2(t))\mathrm{d}w(t), \end{aligned}$$

there does not exist a one-form $\omega(t) \in \mathcal{Z} + \mathcal{U}$ such that $\mathrm{d}y(t+1) - \omega(t) \in \Omega$. Thus the second condition of Theorem 3 is not satisfied and the DDP is not solvable by the static measurement feedback.

5. CONCLUSION

In this paper the notion of the rank of a one-form and the subspace Ω of differential one-forms was used to solve the DDP for nonlinear discrete-time control systems by static measurement feedback. Sufficient conditions for solvability of the DDP were found. Necessary and sufficient conditions were derived from the above conditions for SISO systems and for MISO systems under the additional assumption. The sufficient condition also provided a procedure to find the static measurement feedback to solve the DDP. Because these conditions are very restrictive, further research is necessary. Next step is to extend the results by Xia and Moog (1999) addressing the dynamic measurement feedback in the framework of differential forms for discrete-time systems. Those results can then be compared with those by Kotta et al. (2011), that are obtained using the tools of algebra of functions. Additionally to above theoretical problems the functions in Mathematica will be developed for solving the DDP and integrated into the symbolic software package NLControl, developed in the Institute of Cybernetics at Tallinn University of Technology.

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