Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

of the 18$^{\mathrm{th}}$ International Conference on Process Control

Hotel Titris, Tatranská Lomnica, Slovakia, June 14 – 17, 2011

ISBN 978-80-227-3517-9

http://www.kirp.chtf.stuba.sk/pc11

Editors: M. Fikar and M. Kvasnica

Kowalewski, A., Sokolowski, J.: Sensitivity Analysis of Hyperbolic Optimal Control Systems with Boundary Conditions Involving Time Delays, Editors: Fikar, M., Kvasnica, M., In *Proceedings of the 18th International Conference on Process Control*, Tatranská Lomnica, Slovakia, 531–536, 2011.

Full paper online: http://www.kirp.chtf.stuba.sk/pc11/data/abstracts/017.html

SENSITIVITY ANALYSIS OF HYPERBOLIC OPTIMAL CONTROL SYSTEMS WITH BOUNDARY CONDITIONS INVOLVING TIME DELAYS

A. Kowalewski * and **J. Sokołowski** **

* *Institute of Automatics*
AGH University of Scince and Technology
al. Mickiewicza 30, 30-059 Cracow, Poland
fax: +48 -12 -6341568, e-mail: ako@ia.agh.edu.pl

** *Institut Elie Cartan, Laboratoire de Mathématiques*
Université Henri Poincaré Nancy I
B.P. 239, 54506 Vandoeuvre lés Nancy Cedex, France
e-mail: Jan.Sokolowski@iecn.u-nancy.fr
and
Systems Research Institute of the Polish Academy of Sciences
ul. Newelska 6, 01-447 Warszawa, Poland

Abstract: In the paper the first order sensitivity analysis is performed for a class of optimal control problems for hyperbolic equations with the Neumann boundary conditions involving constant time delays. A singular perturbation of geometrical domain of integration is introduced in the form of a circular hole. The Steklov-Poincaré operator on a circle is defined in order to reduce the problem to regular perturbations in the truncated domain. The optimality system is differentiated with respect to the small parameter and the directional derivative of the optimal control is obtained as a solution to an auxiliary optimal control problem.

Keywords: Sensitivity analysis, hyperbolic system, Neumann boundary condition, time delay.

1. INTRODUCTION

We consider an optimal control problem in the domain with small geometrical defect. The size of the defect is measured by small parameter $\rho > 0$. The presence of the defect results in the singular perturbation of the hyperbolic state equation. Such a perturbation is transformed to the regular perturbation in the truncated domain Ω_R for any $R > \rho > 0$. We perform the sensitivity analysis in the truncated domain using the Steklov-Poincaré operator defined on the circle Γ_R.

The problems of the sensitivity analysis for regular perturbations of optimal control problems were studied in Lasiecka and Sokołowski (1991); Malanowski and Sokołowski (1986); Malanowski (2001); Rao and Sokołowski (2000); Sokołowski (1985 1987 1988); Sokołowski and Zolesio (1992). Singular perturbations of geometrical domains are analysed in Jackowska et al. (2002 2003); Maz'ya et al. (2000); Nazarov (1999); Nazarov and Sokołowski (2004 2003acb); Nazarov et al. (2004); Sokołowski and Żochowski (1999abc 2001 2003). The construction of asymptotic approximation for the Steklov-Poincaré operator is given in Sokołowski and Żochowski (2005).

In particular, in Kowalewski et al. (2010) the sensivity analysis of optimal control problems defined for the wave equation is performed. The small parameter describes the size of an imperfection in the form of a small hole or cavity in the geometrical domain of integration. The initial state equation in the singularly perturbed domain is replaced by the equation in a smooth domain. The imperfection is replaced by its approximation defined by a suitable Steklov's type differential operator. For approximate optimal control problems the well-posedness is shown. One term asymptotics of optimal control are derived and justified for the approximate model. The key role in the arguments is played by the so called "hidden regularity" of boundary traces generated by hyperbolic solutions.

The idea of "hidden regularity" regalarization has been used in the past successfully for boundary control problems, particulary in the context of numerical approximations (Hendrickson and Lasiecka (1993 1995); Lagnese and Leugering (2004); Lasiecka and Triggiani (2000)). Regularizing parameter allows to obtain smooth on the boundary approximations, which can be then taken to appropriate limits. The property of "hidden regularity" is displayed by hyperbolic flows which satisfy the Lopatinski condition (Harmander (1985); Lasiecka et al. (1986); Lasiecka and

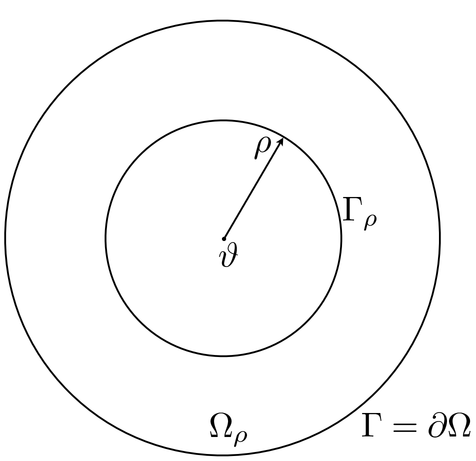

Fig. 1. The domain Ω_ρ in two spatial dimensions.

Triggiani (1990 1991); Sakamoto (1982)). The method of "hidden regularity" regularization has been also applied in domain decomposition procedures introduced and described in Lagnese and Leugering (2004).

In the present paper an optimal control problem in singularly perturbed geometrical domain Ω_ρ is analysed with respect to small parameter $\rho > 0$. We derive the one-term asymptotic expansion of optimal controls. The first term of the expansion, of the order ρ^2 is uniquely determined as an optimal solution to the auxiliary optimal control problem. The control constraints for the auxiliary problem are obtained by an application of the conical differentiability of metric projection in L^2 spaces. Our method is constructive and can lead to numerical procedures for determination of the first order approximations of the optimal controls.

2. PRELIMINARIES

Consider now the distributed parameter system described by the following time delay hyperbolic equation

$$\left.\begin{array}{lll} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = f & \text{in} & \Omega_\rho \times (0,T), \\ \dfrac{\partial y}{\partial \eta} = y(x,t-h) + Gv & \text{on} & \Gamma \times (0,T), \\ \dfrac{\partial y}{\partial \eta} = 0 & \text{on} & \Gamma_\rho \times (0,T), \\ y(x,0) = y_0(x) & \text{in} & \Omega_\rho, \\ \dfrac{\partial y}{\partial t}(x,0) = y_I(x) & \text{in} & \Omega_\rho, \\ y(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0), \end{array}\right\} \quad (1)$$

where:

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad G \in \mathcal{L}(L^2(\Sigma), H^{-5/2}\Xi^{-5/2}(\Sigma)),$$

h is a specified positive number representing a time delay, Ψ_0 is an initial function defined on $\Gamma \times [-h,0)$, $\partial/\partial\eta$ is a normal derivative at $\Gamma\rho$ directed towards the exterior of $\Omega\rho$, $\Omega\rho$ is presented on the Fig. 1.

We denote by

$$\Omega_\rho = \Omega \setminus \overline{B(\rho)} \subset R^2, \quad \partial\, \Omega_\rho = \Gamma \cup \Gamma_\rho, \quad (2)$$

where: Ω is a domain on the plane R^2 with a smooth boundary $\partial\, \Omega$ and

$$B_\rho = \{x : |x - \vartheta| < \rho\} \quad (3)$$

with a smooth boundary $\Gamma\rho$.

First we shall present sufficient conditions for the existence of a unique solution of the problem (1) for the case where the boundary control $v \in L^2(\Sigma)$.

For this purpose, we introduce the space $\mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q)$ (Lions and Magenes (1972), vol. 2, p.131) defined by

$$\begin{array}{l} \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \stackrel{df}{=} \\ \{y | y \in H^{-1,-2}(Q), y'' + Ay \in \Xi^{-3,-3}(Q)\}, \end{array} \quad (4)$$

where: the spaces $H^{-1,-2}(Q)$ and $\Xi^{-3,-3}(Q)$ are defined by (9.5) and (10.4) of Chapter 5 in (Lions and Magenes (1972), vol. 2) respectively. Under the norm of the graph $\mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q)$ is a Hilbert space.

The existence of a unique solution for the mixed initial-boundary value problem (1) on the cylinder Q can be proved using a constructive method, i.e. first solving (1) on the subcylinder Q_1 and in turn on Q_2 etc., until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

For simplicity, we introduce the following notations:

$$\left.\begin{array}{l} Q = \Omega_\rho \times (0,T) \\ \Sigma = \Gamma \times (0,T) \\ E_j \stackrel{\triangle}{=} ((j-1)h, jh) \\ Q_j = \Omega_\rho \times E_j \\ \Sigma_j = \Gamma \times E_j \\ \Sigma_0 = \Gamma \times [-h,0) \end{array}\right\} \text{ for } \quad j = 1, ..., K. \quad (5)$$

Using Theorem 10.1 of (Lions and Magenes (1972), vol. 2, p. 132) we can prove the following result.

Theorem 1. Let y_0, y_I, Ψ_0, v and f be given with $y_0 \in \Xi^{-3/2}(\Omega)$, $y_I \in \Xi^{-5/2}(\Omega)$,
$\Psi_0 \in H^{-5/2}\Xi^{-5/2}(\Sigma_0)$, $v \in L^2(\Sigma)$ and $f \in \Xi^{-3,3}(Q)$. Then there exists a unique solution $y \in \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q)$ for the problem (1). Moreover, $y(\cdot, jh) \in \Xi^{-3/2}(\Omega)$ and $\dfrac{\partial y}{\partial t}(\cdot, jh) \in \Xi^{-5/2}(\Omega)$ for $j = 1, ...K$.

The spaces appearing in the Theorem 1 are defined in Lions and Magenes (1972).

Let us surround Γ_ρ by the circle Γ_R such that $R > \rho > 0$ (Fig. 2) .

Consequently, we denote

$$\Omega_R = \Omega \setminus \overline{B(R)}, \quad (6)$$

where:

$$B(R) = \{x : |x - \vartheta| < R\}. \quad (7)$$

We set the non-local Neumann boundary condition on Γ_R:

$$\frac{\partial y}{\partial \eta} = A_\rho(y) \text{ on } \Gamma_R, \quad (8)$$

where: A_ρ is a Steklov-Poincare operator defined in the domain $C(R,\rho) = B(R) \setminus \overline{B(\rho)}$.The operator A_ρ is a

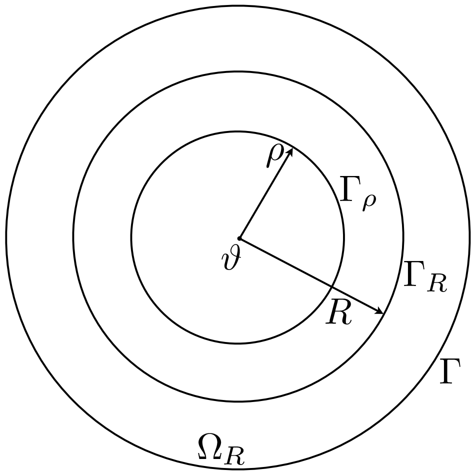

Fig. 2. The domain Ω_R.

mapping of $H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$. Consequently, we consider in $\Omega_R \times (0,T)$ the following time delay hyperbolic equation:

$$\left.\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} - \Delta y = f && \text{in} \quad \Omega_R \times (0,T),\\
&\frac{\partial y}{\partial \eta} = y(x,t-h) + Gv && \text{on} \quad \Gamma \times (0,T),\\
&\frac{\partial y}{\partial \eta} = A_\rho(y) && \text{on} \quad \Gamma_R \times (0,T),\\
&y(x,0) = y_0(x) && \text{in} \quad \Omega_R.\\
&\frac{\partial y}{\partial t}(x,0) = y_I(x) && \text{in} \quad \Omega_R,\\
&y(x,t') = \Psi_0(x,t') && \text{in} \quad \Gamma \times [-h,0),
\end{aligned}\right\} \tag{9}$$

We shall investigate the dependence of optimal solutions on the small parameter $\rho > 0$.

The small hole $B(\rho)$ is a singular perturbation in the domain $\Omega\rho$. Consequently, the same small hole constitutes regular perturbation in the domain Ω_R.

Using the results of Sokołowski and Żochowski (2005) we obtain the following expansion for the operator A_ρ:

$$\begin{aligned}
&A_\rho = A_0 + \rho^2 B + O(\rho^4)\\
&\text{in the operator norm}\\
&\mathcal{L}(H^{1/2}(\Gamma_R), H^{-1/2}(\Gamma_R)),
\end{aligned} \tag{10}$$

where: the remainder $O(\rho^4)$ is uniformly bounded on bounded sets in the space $H^{1/2}(\Gamma_R)$.

Corollary 1. In the space $\mathcal{D}^{-1}_{A+\mathcal{D}^2_t}(Q)$ the solution of the hyperbolic equation (for $\rho = 0$) can be represented as

$$\left.\begin{aligned}
&\frac{\partial^2 y^0}{\partial t^2} - \Delta y^0 = f && \text{in} \quad \Omega_R \times (0,T),\\
&\frac{\partial y^0}{\partial \eta} = y^0(x,t-h) + Gv && \text{on} \quad \Gamma \times (0,T),\\
&\frac{\partial y^0}{\partial \eta} = A_0(y^0) && \text{on} \quad \Gamma_R \times (0,T),\\
&y^0(x,0) = y_0(x) && \text{in} \quad \Omega_R,\\
&\frac{\partial y^0}{\partial t}(x,0) = y_I(x) && \text{in} \quad \Omega_R,\\
&y^0(x,t') = \Psi_0(x,t') && \text{in} \quad \Gamma \times [-h,0).
\end{aligned}\right\} \tag{11}$$

We shall look the expansion of the solution y^ρ in $\Omega_R \times (0,T)$:

$$\begin{aligned}
y^\rho &= y^0 + \rho^2 y^1 + \tilde{y} =\\
&= y^0 + \rho^2 y^1 + \rho^4 \hat{y}
\end{aligned} \tag{12}$$

Consequently, the Neumann boundary condition in (9) can be rewritten as

$$\begin{aligned}
\frac{\partial y^\rho}{\partial \eta} &= A_\rho(y^\rho) =\\
&= A_0(y^\rho) + \rho^2 B(y^\rho) + \rho^4 \tilde{A}(y^\rho)
\end{aligned} \tag{13}$$

Substituting (12) into (13) we obtain

$$\begin{aligned}
&\frac{\partial y^0}{\partial \eta} + \rho^2 B \frac{\partial y^1}{\partial \eta} + \frac{\partial \tilde{y}}{\partial \eta} =\\
&= A_0(y^0 + \rho^2 y^1 + \tilde{y}) +\\
&+ \rho^2 B(y^0 + \rho^2 y^1 + \tilde{y}) + \rho^4 \tilde{A}(y^\rho)
\end{aligned} \tag{14}$$

Comparing components with the same powers we get

$$\left.\begin{aligned}
&\rho^0 : \frac{\partial y^0}{\partial \eta} = A_0(y^0)\\
&\rho^2 : \rho^2 \frac{\partial y^1}{\partial \eta} = \rho^2 [A_0 y^1 + B y^0]
\end{aligned}\right\} \tag{15}$$

Hence it follows the following expansion of solutions:

Let us denote by y^0 the solution of the problem (11) corresponding to a given parameter $\rho = 0$.

Subsequently, y^1 corresponding to a given parameter ρ^2 is a solution of the following equation:

$$\left.\begin{aligned}
&\frac{\partial^2 y^1}{\partial t^2} - \Delta y^1 = 0 && \text{in} \quad \Omega_R \times (0,T),\\
&\frac{\partial y^1}{\partial \eta} = y^1(x,t-h) + Gv && \text{on} \quad \Gamma \times (0,T),\\
&\frac{\partial y^1}{\partial \eta} = A_0(y^1) + B(y^0) && \text{on} \quad \Gamma_R \times (0,T),\\
&y^1(x,0) = 0 && \text{in} \quad \Omega_R,\\
&\frac{\partial y^1}{\partial t}(x,0) = 0 && \text{in} \quad \Omega_R,\\
&y^1(x,t') = \Psi_0(x,t') && \text{in} \quad \Gamma \times [-h,0).
\end{aligned}\right\} \tag{16}$$

3. PROBLEM FORMULATION. OPTIMIZATION THEOREM.

We shall now consider the optimal boundary control problem in domains Ω_ρ and Ω_R respectively. Let us denote by $U = L^2(\Gamma \times (0,T))$ the space of controls. The time horizon T is fixed in our problem.

Let us consider in $\Omega_\rho \times (0,T)$ the following time delay hyperbolic equation

$$\left.\begin{array}{lll} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = f & \text{in} & \Omega_\rho \times (0,T), \\ & \text{supp} & f \subset \Omega_R \times (0,T), \\ \dfrac{\partial y}{\partial \eta} = y(x,t-h) + Gv & \text{on} & \Gamma \times (0,T), \\ \dfrac{\partial y}{\partial \eta} = 0 & \text{on} & \Gamma_\rho \times (0,T), \\ y(x,0) = y_0(x) & \text{in} & \Omega_\rho \\ & \text{supp} & y_0 \subset \Omega_R, \\ \dfrac{\partial y}{\partial t}(x,0) = y_I(x) & \text{in} & \Omega_\rho, \\ & \text{supp} & y_I \subset \Omega_R, \\ y(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0). \end{array}\right\} \quad (17)$$

The performance functional is given by

$$I(v) = \frac{1}{2}\left\| y(v) - z_d \right\|^2_{H^{-1,-2}(\Omega_R \times (0,T))} + \frac{\alpha}{2}\left\| v \right\|^2_{L^2(\Gamma \times (0,T))}. \quad (18)$$

Finally, we assume the following constraints on the control $v \in U_{ad}$:

$$U_{ad} = \{v \in L^2(\Gamma \times (0,T)), 0 \le v(x,t) \le 1\}. \quad (19)$$

Subsequently, we consider in $\Omega_R \times (0,T)$ the following hyperbolic time delay equation

$$\left.\begin{array}{lll} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = f & \text{in} & \Omega_R \times (0,T), \\ \dfrac{\partial y}{\partial \eta} = y(x,t-h) + Gv & \text{on} & \Gamma \times (0,T), \\ \dfrac{\partial y}{\partial \eta} = A_\rho(y) & \text{on} & \Gamma_R \times (0,T), \\ y(x,0) = y_0(x) & \text{in} & \Omega_R, \\ \dfrac{\partial y}{\partial t}(x,0) = y_I(x) & \text{in} & \Omega_R, \\ y(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0). \end{array}\right\} \quad (20)$$

The performance functional and constraints on the control are given by (18) and (19).

Result: The Solution of the problem (20) (in the domain Ω_R) is a restriction of the solution of the problem (17) (in the domain Ω_ρ) to Ω_R. Hence, we have the possibility of replacing the singular perturbation of the domain $B(\rho)$ by the regular perturbation on the boundary Γ_R in a smaller domain Ω_R. Consequently, we shall analyse the optimal boundary control problem (18)-(20) in the domain Ω_R. Moreover, we assume the fixed parameter $\rho > 0$. The solving of the formulated optimal control problem is equivalent to seeking a $v_0 \in U_{ad}$ such that $I(v_0) \le I(v)$ $\forall v \in U_{ad}$.

From Lions' scheme (Theorem 1.3 Lions (1971), p. 10) it follows that for $\alpha > 0$ a unique optimal control v_0 is characterized by the following condition

$$I'(v_0)(v - v_0) \ge 0 \quad \forall v \in U_{ad}. \quad (21)$$

Using the form of the performance functional (18) we can express (21) in the following form:

$$\Big\langle (y(v_0) - z_d, y(v) - y(v_0) \Big\rangle_{H^{-1,-2}(\Omega_R \times (0,T))} + \alpha \Big\langle v_0, v - v_0 \Big\rangle_{L^2(\Gamma \times (0,T))} \ge 0 \quad \forall v \in U_{ad}. \quad (22)$$

To simplify (22), we introduce the adjoint equation and for every $v \in U_{ad}$. we define the adjoint variable $p = p(v) = p(x,t;v)$ as the solution of the following equation

$$\left.\begin{array}{ll} \dfrac{\partial^2 p}{\partial t^2} - \Delta p = y(v) - z_d & \text{in } \Omega_R \times (0,T), \\ \dfrac{\partial p}{\partial \eta} = p(x,t+h) & \text{on } \Gamma \times (0,T-h), \\ \dfrac{\partial p}{\partial \eta} = 0 & \text{on } \Gamma \times (T-h,T), \\ \dfrac{\partial p}{\partial \eta} = A_\rho(p) & \text{on } \Gamma_R \times (0,T), \\ p(x,T;v) = 0 & \text{in } \Omega_R, \\ p'(x,T;v) = 0 & \text{in } \Omega_R. \end{array}\right\} \quad (23)$$

Theorem 2. Let the hypothesis of Theorem 1 be satisfied. Then for given $z_d \in H^{-1,-2}(\Omega_R \times (0,T))$ and any $v_0 \in L^2(\Sigma)$, there exists a unique solution $p(v_0) \in H^{3,3}(\Omega_R \times (0,T)) \subset \Xi^{3,3}(\Omega_R \times (0,T))$ for the problem (23).

We simplify (22) using the adjoint equation (23). Consequently, after transformations we obtain the following formula

$$\Big\langle G^* p + \alpha\, v_0, v - v_0 \Big\rangle_{L^2(\Gamma \times (0,T))} \ge 0 \quad \forall v \in U_{ad}. \quad (24)$$

Theorem 3. For the problem (20) with the performance functional (18) with $\alpha > 0$, and with constraints on the control (19), there exists a unique optimal control v_0 which satisfies the maximum condition (24). Moreover, $v_0 = P_{U_{ad}}\left(-\frac{1}{\alpha}G^* p\right)$ where $P_{U_{ad}}$ is a projective operator.

4. THE SENSITIVITY OF OPTIMAL CONTROLS

Theorem 4. We have the following expansion of the optimal control in $L^2(\Gamma \times (0,T))$, with respect to the small parameter,

$$v_\rho = v_0 + \rho^2 q + o(\rho^2) \quad (25)$$

for $\rho > 0$.

Moreover, we assume that ρ is a sufficiently small. The function q in (25) is a optimal solution of the following optimal control problem:

The state equation

$$\left.\begin{array}{ll} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = 0 & \text{in } \Omega_R \times (0,T), \\ \dfrac{\partial w}{\partial \eta} = w(x,t-h) + Gq & \text{on } \Gamma \times (0,T), \\ \dfrac{\partial w}{\partial \eta} = A_0(w) + B(y^0) & \text{on } \Gamma_R \times (0,T), \\ w(x,0) = 0 & \text{in } \Omega_R, \\ \dfrac{\partial w}{\partial t}(x,0) = 0 & \text{in } \Omega_R, \\ w(x,t') = \Psi_0(x,t') & \text{on } \Gamma \times [-h,0), \end{array}\right\} \tag{26}$$

where: $w = y^1$.

The performance functional

$$I(u) = \frac{1}{2}\left\|w(q)\right\|^2_{H^{-1,-2}(\Omega_R \times (0,T))} + \frac{\alpha}{2}\left\|u\right\|^2_{L^2(\Gamma \times (0,T))}. \tag{27}$$

The adjoint equation

$$\left.\begin{array}{ll} \dfrac{\partial^2 z}{\partial t^2} - \Delta z = w(q) & \text{in } \Omega_R \times (0,T), \\ \dfrac{\partial z}{\partial \eta} = z(x,t+h) & \text{on } \Gamma \times (0,T-h), \\ \dfrac{\partial z}{\partial \eta} = 0 & \text{on } \Gamma \times (T-h,T), \\ \dfrac{\partial z}{\partial \eta} = A_0(z) + B(p^0) & \text{on } \Gamma_R \times (0,T), \\ z(x,T) = 0 & \text{in } \Omega_R, \\ z'(x,T) = 0 & \text{in } \Omega_R, \end{array}\right\} \tag{28}$$

where: $z = p^1$.

Then, the optimal control q is characterized by

$$\left\langle w(q), w(u) - w(q)\right\rangle_{H^{-1,-2}(\Omega_R \times (0,T))} + \alpha\left\langle q, u - q\right\rangle_{L^2(\Gamma \times (0,T))} \geq 0 \quad \forall v \in U_{ad}, \tag{29}$$

where: S_{ad} is a set of admissible controls such that

$$\begin{array}{l} S_{ad} = \Big\{u \in L^2(\Gamma \times (0,T))\Big| \\ u(x,t) \geq 0 \text{ on the set} \\ \qquad E_0 = \{(x,t)|v_0(x,t) = 0\}, \\ u(x,t) < 0 \text{ on the set} \\ \qquad E_1 = \{(x,t)|v_0(x,t) = 1\}, \\ \left\langle G^* p_0 + \alpha\, v_0, u\right\rangle_{L^2(\Gamma \times (0,T))} = 0\}, \end{array} \tag{30}$$

where:

p_0 is a adjoint state for $\rho = 0$,

v_0 is a optimal solution for $\rho = 0$ such that

$0 \leq v_0(x,t) \leq 1$.

We simplify (29) using the adjoint equation (28). After transformations we obtain the following maximum condition

$$\left\langle G^* z + \alpha\, q, u - q\right\rangle_{L^2(\Gamma \times (0,T))} \geq 0 \quad \forall u \in S_{ad}. \tag{31}$$

Theorem 5. For the time delay hyperbolic problem

$$\left.\begin{array}{ll} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = 0 & \text{in } \Omega_R \times (0,T), \\ \dfrac{\partial w}{\partial \eta} = w(x,t-h) + Gu & \text{on } \Gamma \times (0,T), \\ \dfrac{\partial w}{\partial \eta} = A_0(w) + B(y^0) & \text{on } \Gamma_R \times (0,T), \\ w(x,0) = 0 & \text{in } \Omega_R, \\ \dfrac{\partial w}{\partial t}(x,0) = 0 & \text{in } \Omega_R, \\ w(x,t') = \Psi_0(x,t') & \text{in } \Gamma \times [-h,0), \end{array}\right\} \tag{32}$$

with the performance functional (27) with $\alpha > 0$, and with constraints on the control (30), there exists a unique optimal control q which satisfies the maximum condition (31).

5. CONCLUSIONS

The results presented in the paper can be treated as a generalization of the results obtained in Sokołowski and Żochowski (2005) onto the case of hyperbolic systems with boundary condition involving time delays.

In this paper we have considered the mixed initial boundary value problems of hyperbolic type.

We can also consider similar optimal control problems for parabolic-hyperbolic systems.

The ideas mentioned above will be developed in forthcoming papers.

ACKNOWLEDGEMENTS

The research presented here was carried out within the research programme AGH University of Science and Technology, No. 11.11.120.768.

REFERENCES

Harmander, L. (1985). *The Analysis of Linear Partial Differential Operators - Vol.III.* Springer-Verlag, Berlin-Heidelberg.

Hendrickson, E. and Lasiecka, I. (1993). Numerical approximations and regularizations of Ricati equations arising in hyperbolic dynamics with unbounded control operators. *Computational Optimization and Applications*, 2, 343–390.

Hendrickson, E. and Lasiecka, I. (1995). Finite dimensional approximations of boundary control problems arising in partially observed hyperbolic systems. *Dynamics of Continuous Discrete and Impulsive Systems*, 1, 101–142.

Jackowska, L., Sokołowski, J., and Żochowski, A. (2003). Topological optimization and inverse problems. *Computer Assisted Mechanics and Engineering Sciences*, 10, 163–176.

Jackowska, L., Sokołowski, J., Żochowski, A., and Henrot, A. (2002). On numerical solution of shape inverse problems. *Computational Optimization and Applications*, 23, 231–255.

Kowalewski, A., Lasiecka, I., and Sokołowski, J. (2010). Sensitivity analysis of hyperbolic optimal control problems (published on-line: 20 November 2010). *Computational Optimization and Applications (*to appear).

Lagnese, J. and Leugering, G. (2004). *Domain Decomposition Methods in Optimal Control of Partial Differential Equations*. Birkhäuser, Basel.

Lasiecka, I., Lions, J., and Triggiani, R. (1986). Nonhomogeneous boundary value problems for second order hyperbolic operators. *Journal de Mathematiques Pures et Appliquees*, 65, 149–192.

Lasiecka, I. and Sokołowski, J. (1991). Sensitivity analysis of constrained optimal control problem for wave equation. *SIAM Journal on Control and Optimization*, 29, 1128–1149.

Lasiecka, I. and Triggiani, R. (1990). Sharp regularity results for second order hyperbolic equations of Neumann type. *Annali di Matematica Pura ed Applicata*, CLVII, 1128–1149.

Lasiecka, I. and Triggiani, R. (1991). Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions. *Journal of Differetial Equations*, 94, 112–164.

Lasiecka, I. and Triggiani, R. (2000). *Control Theory for Partial Differential Equations*. Cambridge University Press, Cambridge.

Lions, J. (1971). *Optimal Control of Systems Governed by Partial Differential Equations*. Springer-Verlag, Berlin-Heidelberg.

Lions, J. and Magenes, E. (1972). *Non-Homogeneous Boundary Value Problems and Applications –Vols. 1 adn. 2*. Springer-Verlag, Berlin-Heidelberg.

Malanowski, K. (2001). *Stability and sensitivity analysis for optimal control problems with control-state constraints*. Disertations Math. (Rozprawy Mat.), Warsaw.

Malanowski, K. and Sokołowski, J. (1986). Sensitivity of solutions to convex, control constrained optimal control problems for distributed parameter systems. *Journal of Mathematical Analysis and Applications*, 120, 240–263.

Maz'ya, V., Nazarov, S., and Plamenevskij, B. (2000). *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains -vol.1*. Birkhäuser Verlag, Basel.

Nazarov, S.A. (1999). Asymptotic conditions at a point, self adjoint extensions of operators and the method of matched asymptotic expansions. *American Mathematical Society Translations (2)*, 198, 77–125.

Nazarov, S.A., Slutskij, S.A., and Sokołowski, J. (2004). Topological derivative of the energy functional due to formation of a thin ligament on a spatial body. *Les prépublications de l'Institut Élie Cartan*, 14.

Nazarov, S.A. and Sokołowski, J. (2003a). Asymptotic analysis of shape functionals. *Journal de Mathématiques pures et appliquées*, 82, 125–196.

Nazarov, S.A. and Sokołowski, J. (2003b). Self adjoint extensions for the neumann laplacian in application to shape optimization. *Les prépublications de l'Institut Élie Cartan*, 9.

Nazarov, S.A. and Sokołowski, J. (2003c). Self adjoint extensions of differential operators in application to shape optimization. *Comptes Rendus Mecanique*, 331, 667–672.

Nazarov, S.A. and Sokołowski, J. (2004). The topological derivative of the dirichlet integral due to formation of a thin ligament. *Siberian Math. J.*, 45, 341–355.

Rao, M. and Sokołowski, J. (2000). Tangent sets in banach spaces and applications to variational inequalities. *Les prépublications de l'Institut Élie Cartan*, 42.

Sakamoto, R. (1982). *Hyperbolic Boundary Value Problems*. Cambridge University Press, Cambridge.

Sokołowski, J. (1985). Differential stability of solutions to constrained optimization problems. *Appl. Math. Optim.*, 13, 97–115.

Sokołowski, J. (1987). Sensitivity analysis of control constrained optimal control problems for distributed parameter systems. *SIAM J. Control and Optimization*, 25, 1542–1556.

Sokołowski, J. (1988). Shape sensitivity analysis of boundary optimal control problems for parabolic systems. *SIAM J. Control and Optimization*, 26, 763–787.

Sokołowski, J. and Żochowski, A. (1999a). On topological derivative in shape optimization. *SIAM J. Control and Optimization*, 37, 1251–1272.

Sokołowski, J. and Żochowski, A. (1999b). Topological derivative for optimal control problems. *Control and Cybernetics*, 28, 611–626.

Sokołowski, J. and Żochowski, A. (1999c). Topological derivatives for elliptic problems. *Inverse Problems*, 1, 123–134.

Sokołowski, J. and Żochowski, A. (2001). Topological derivatives of shape functionals for elasticity systems. *Mechanics of Structures and Machines*, 29, 333–351.

Sokołowski, J. and Żochowski, A. (2003). Optimality conditions for simultaneous topology and shape optimization. *SIAM J. Control and Optimization*, 42, 1198–1221.

Sokołowski, J. and Żochowski, A. (2005). Topological derivatives for obstacle problems. *Les prépublications de l'Institut Élie Cartan*, 12.

Sokołowski, J. and Zolesio, J.P. (1992). *Introduction to Shape Optimization. Shape Sensitivity Analysis*. Springer Verlag, Berlin-Heidelberg.