Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

of the 18$^{\mathrm{th}}$ International Conference on Process Control

Hotel Titris, Tatranská Lomnica, Slovakia, June 14 – 17, 2011

ISBN 978-80-227-3517-9

http://www.kirp.chtf.stuba.sk/pc11

Editors: M. Fikar and M. Kvasnica

H∞ control of time-delay systems with time-varying delays

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Abstract: The linear matrix inequality (LMI) based memory-less controller design approach for continuous time systems with time-varying delays is presented in the paper. If the time-delay variation is from the specified range the design conditions are formulated as feasibility problem and expressed over a set of LMIs with the matrix rank constraints implying from integral quadratic constraints. The proposed method is demonstrated using a system model example.

Keywords: Linear matrix inequality, time-delay systems, time-varying delays, linear systems.

1. INTRODUCTION

Continuous-time control systems are used in many industrial applications, where time delays can take a deleterious effect on both the stability and the dynamic performance in the open and closed-loop systems. Thus, the problems of asymptotic stability and stabilization for time-delay systems have received considerable attention and intensive activity are done to develop a sophisticated control for such systems.

Linear matrix inequality (LMI) approaches based on convex optimization algorithms have been extensively applied to solve the above mentioned problem, since it can be solved numerically efficiently by using interior-point algorithm which has recently been developed for solving optimization problem. Using the LMI approach, two categories of stability criteria for guaranteeing stability of the delayed system were developed. Delay independent criteria provide conditions which guarantee stability for any length of the time delay, whereas delay dependent criteria exploit a priori knowledge of upper-bounds on the amount of time-delay or its variation. Of course, delay dependent criteria are generally less conservative than delay-independent ones since more information about the time-delay is assumed to be known.

The use of Lyapunov method for the stability analysis of the time delay systems has been ever growing subject of interest starting with the pioneering works of Krasovskii (Krasovskii (1956), Krasovskii (1963)). Progress review in this research field is presented e.g. in Niculescu at al. (1998), Wu at al. (2004), Kao and Rantzer (2005), and the references therein, some special forms of Lyapunov-Krasovskii functions can be also found in Wu at al. (2010).

Considering the influence of time-varying delay as perturbation in the system, where delay parameter is an unknown time-varying function with given upper bounds on the magnitude and the variation, the paper address the problems of asymptotic stabilization for such time-delay systems if the time-delay variation is from the specified range. Translating into LMI framework the closed-loop system stability is characterized in the terms of convex LMIs, where the convex parameterizations are based on extended Lyapunov function with integral quadratic constraints in the bounded real lemma form.

2. PROBLEM DESCRIPTION

Through this paper the task is concerned with the computation of a state feedback $\boldsymbol{u}(t)$, which control the time-delay linear dynamic system given by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{A}_d\boldsymbol{q}(t-\tau(t)) + \boldsymbol{B}\boldsymbol{u}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) + \boldsymbol{D}\boldsymbol{u}(t) \tag{2}$$

with initial condition

$$\boldsymbol{q}(\vartheta) = \varphi(\vartheta),\ \forall\vartheta \in \langle -h, 0\rangle \tag{3}$$

where $\tau(t)$ is an unknown time-varying parameter satisfying conditions

$$0 \le \tau(t) \le h, \qquad |\,\dot{\tau}(t)\,| \le d, \quad \forall t \ge 0 \tag{4}$$

where $\boldsymbol{q}(t) \in I\!R^n$ stands up for the system state, $\boldsymbol{u}(t) \in I\!R^r$ denotes the control input, $\boldsymbol{y}(t) \in I\!R^m$ is the system measurable output, and nominal system matrices $\boldsymbol{A} \in I\!R^{n\times n}$, $\boldsymbol{A}_d \in I\!R^{n\times n}$, $\boldsymbol{B} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$ and $\boldsymbol{D} \in I\!R^{m\times r}$ are real matrices.

Problem of the interest is to design asymptotically stable closed-loop system with the linear memoryless state feedback controller of the form

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}(t) \tag{5}$$

for $t \ge 0$, where matrix $\boldsymbol{K} \in I\!R^{r\times n}$ is a gain matrix.

3. BASIC PRELIMINARIES

Proposition 1. (Bounded real lemma) System (1), (2), where $\boldsymbol{A}_d = \boldsymbol{0}$, is stable with quadratic performance $\|\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}\|_\infty \leq \gamma$ if there exist a symmetric positive definite matrix $\boldsymbol{P} > 0$ and a positive scalar $\gamma > 0$ such that

$$\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{6}$$

where $\boldsymbol{I}_r \in \mathbb{R}^{r\times r}$, $\boldsymbol{I}_m \in \mathbb{R}^{m\times m}$ are identity matrices, respectively,

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Generally, there exists an enough small $\gamma > 0$ such that Lyapunov function can be defined as follows

$$\begin{aligned} v(\boldsymbol{q}(t)) &= \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t)+ \\ &+\int_0^t(\boldsymbol{y}^T(r)\boldsymbol{y}(r)-\gamma^2\boldsymbol{u}^T(r)\boldsymbol{u}(r))\mathrm{d}r > 0 \end{aligned} \tag{7}$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, $\gamma > 0 \in \mathbb{R}$, and evaluating the derivative of $v(\boldsymbol{q}(t))$ with respect to t along a system trajectory then it yields

$$\begin{aligned} \dot{v}(\boldsymbol{q}(t)) &= \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t)+\boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t)+ \\ &+\boldsymbol{y}^T(t)\boldsymbol{y}(t)-\gamma^2\boldsymbol{u}^T(t)\boldsymbol{u}(t) < 0 \end{aligned} \tag{8}$$

Thus, substituting (1), (2) into (8) it can be written

$$\begin{aligned} \dot{v}(\boldsymbol{q}(t)) &= (\boldsymbol{A}\boldsymbol{q}(t)+\boldsymbol{B}\boldsymbol{u}(t))^T\boldsymbol{P}\boldsymbol{q}(t)+ \\ &+\boldsymbol{q}^T(t)\boldsymbol{P}(\boldsymbol{A}\boldsymbol{q}(t)+\boldsymbol{B}\boldsymbol{u}(t))-\gamma\boldsymbol{u}^T(t)\boldsymbol{u}(t)+ \\ &+(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t))^T(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t)) < 0 \end{aligned} \tag{9}$$

and with the next notation

$$\boldsymbol{q}_c^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) \end{bmatrix} \tag{10}$$

it is obtained

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}_c^T(t)\boldsymbol{P}_c\boldsymbol{q}_c(t) < 0 \tag{11}$$

where

$$\boldsymbol{P}_c = \begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B} \\ * & -\gamma^2\boldsymbol{I}_r \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} < 0 \tag{12}$$

Since

$$\begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^T \\ \boldsymbol{D}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \geq 0 \tag{13}$$

Schur complement property implies

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{C}^T \\ * & \boldsymbol{0} & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} \geq 0 \tag{14}$$

and using (14) the LMI condition (12) can be written compactly as (6). This concludes the proof. ∎

Proposition 2. (Symmetric upper-bound inequality) Let $f(\boldsymbol{x}(\eta))$, $\boldsymbol{x}(\eta) \in \mathbb{R}^n$, $\boldsymbol{X} > 0$, $\boldsymbol{X} \in \mathbb{R}^{n\times n}$ is a real positive definite and integrable vector function of the form

$$f(\boldsymbol{x}(\eta)) = \boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta) \tag{15}$$

such, that there exists a well defined integration as following

$$\int_{t-h}^t f(\boldsymbol{x}(\eta))\mathrm{d}\eta > 0 \tag{16}$$

with $h > 0$, $h \in \mathbb{R}$, then

$$\int_{t-h}^t \boldsymbol{x}^T(\eta)\mathrm{d}\eta\boldsymbol{X}\int_{t-h}^t \boldsymbol{x}(\eta)\mathrm{d}\eta \leq h\int_{t-h}^t \boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta)\mathrm{d}\eta \tag{17}$$

Proof. Since for (15) it can be written

$$\boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta)-\boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta) = 0 \tag{18}$$

and according to Schur complement property it is true that

$$\begin{bmatrix} \boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta) & \boldsymbol{x}^T(\eta) \\ \boldsymbol{x}(\eta) & \boldsymbol{X}^{-1} \end{bmatrix} = 0 \tag{19}$$

then the integration of (19) with respect to η gives

$$\begin{bmatrix} \int_{t-h}^t \boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta)\mathrm{d}\eta & \int_{t-h}^t \boldsymbol{x}^T(\eta)\mathrm{d}\eta \\ * & \int_{t-h}^t \boldsymbol{X}^{-1}\mathrm{d}\eta \end{bmatrix} \geq 0 \tag{20}$$

$$\begin{bmatrix} \int_{t-h}^t \boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta)\mathrm{d}\eta & \int_{t-h}^t \boldsymbol{x}^T(\eta)\mathrm{d}\eta \\ * & h\boldsymbol{X}^{-1} \end{bmatrix} \geq 0 \tag{21}$$

respectively. Thus,

$$h^{-1}\int_{t-h}^t \boldsymbol{x}^T(\eta)\mathrm{d}\eta\boldsymbol{X}\int_{t-h}^t \boldsymbol{x}(\eta)\mathrm{d}\eta \leq \int_{t-h}^t \boldsymbol{x}^T(\eta)\boldsymbol{X}\boldsymbol{x}(\eta)\mathrm{d}\eta \tag{22}$$

and it is evident that with $h > 0$ (22) implies (17). This concludes the proof. ∎

4. DESCRIPTOR SYSTEM PROPERTIES

Adding and subtracting vector element $\boldsymbol{A}_d\boldsymbol{q}(t)$ to (1) results in

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{B}\boldsymbol{u}(t)+(\boldsymbol{A}+\boldsymbol{A}_d)\boldsymbol{q}(t)-\boldsymbol{A}_d(\boldsymbol{q}(t)-\boldsymbol{q}(t-\tau(t))) \tag{23}$$

It is well-known fact that the descriptor model (23) is not equivalent to system (1), since this transformation introduces additional dynamics. However, stability of system (23) does imply stability of system (1), i.e. the delay-derivative-independent stability criterion it is necessary to be stated.

Considering $\boldsymbol{u}(t)=\mathbf{0}$ then the autonomous system to (23) can be written as

$$\dot{\boldsymbol{q}}(t)=(\boldsymbol{A}+\boldsymbol{A}_d)\boldsymbol{q}(t)+\boldsymbol{A}_d\boldsymbol{u}^{\circ}(t) \tag{24}$$

where

$$\boldsymbol{u}^{\circ}(t)=-(\boldsymbol{q}(t)-\boldsymbol{q}(t-\tau(t))) \tag{25}$$

$$\boldsymbol{u}^{\circ}(t)=-\boldsymbol{I}\int\limits_{t-\tau(\eta)}^{t}\dot{\boldsymbol{q}}(\eta)\mathrm{d}\eta=-\boldsymbol{I}\int\limits_{t-\tau(\eta)}^{t}\boldsymbol{y}^{\circ}(\eta)\mathrm{d}\eta \tag{26}$$

respectively, with

$$\boldsymbol{y}^{\circ}(t)=\dot{\boldsymbol{q}}(t)=(\boldsymbol{A}+\boldsymbol{A}_d)\boldsymbol{q}(t)+\boldsymbol{A}_d\boldsymbol{u}^{\circ}(t) \tag{27}$$

Therefore, (24), (25) can be interpreted as a dynamic system with uncertain internal integral closed-loop feedback.

Denoting

$$\dot{\boldsymbol{q}}(t)=\boldsymbol{A}^{\circ}\boldsymbol{q}(t)+\boldsymbol{B}^{\circ}\boldsymbol{u}^{\circ}(t) \tag{28}$$

$$\boldsymbol{y}^{\circ}(t)=\boldsymbol{C}^{\circ}\boldsymbol{q}(t)+\boldsymbol{D}^{\circ}\boldsymbol{u}^{\circ}(t) \tag{29}$$

where

$$\boldsymbol{A}^{\circ}=\boldsymbol{C}^{\circ}=\boldsymbol{A}+\boldsymbol{A}_d,\quad \boldsymbol{B}^{\circ}=\boldsymbol{D}^{\circ}=\boldsymbol{A}_d \tag{30}$$

then an equivalent Lyapunov function to the (7) can be introduced. Unlike a delay-free linear system there exist state boundaries in the descriptor system, so the weighting matrices of Lyapunov function have to be introduced in special forms.

Considering the quadratic integral form

$$\begin{aligned}J_1&=\int\limits_0^{\infty}\boldsymbol{u}^{\circ T}(t)\boldsymbol{X}\boldsymbol{u}^{\circ}(t)\mathrm{d}t=\\&=\int\limits_0^{\infty}\int\limits_{t-\tau(\eta)}^{t}\dot{\boldsymbol{q}}^T(\eta)\mathrm{d}\eta\,\boldsymbol{X}\int\limits_{t-\tau(\eta)}^{t}\dot{\boldsymbol{q}}(\eta)\mathrm{d}\eta\mathrm{d}t\end{aligned} \tag{31}$$

then using (17) it is obvious that

$$\begin{aligned}J_1&\le\int\limits_0^{\infty}\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(\eta)\mathrm{d}\eta\,\boldsymbol{X}\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}(\eta)\mathrm{d}\eta\mathrm{d}t\le\\&\le\int\limits_0^{\infty}h\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(\eta)\boldsymbol{X}\dot{\boldsymbol{q}}(\eta)\mathrm{d}\eta\mathrm{d}t=h^2\int\limits_0^{\infty}\dot{\boldsymbol{q}}^T(\eta)\boldsymbol{X}\dot{\boldsymbol{q}}(\eta)\mathrm{d}\eta\end{aligned} \tag{32}$$

It is evident that the integral norm-weighting matrix in (32) is independent of d. Analogously, respecting

$$J_2=\int\limits_0^{\infty}\boldsymbol{q}^T(t-\tau(t))\boldsymbol{X}\boldsymbol{q}(t-\tau(t))\mathrm{d}t \tag{33}$$

then setting

$$t-\tau(t)=\eta,\quad (1-\dot{\tau}(t))\mathrm{d}t=\mathrm{d}\eta \tag{34}$$

(33) can be rewritten as follows

$$\begin{aligned}J_2&=\int\limits_{-\tau(0)}^{\infty}\frac{1}{1-\dot{\tau}(\eta(t))}\boldsymbol{q}^T(\eta)\boldsymbol{X}\boldsymbol{q}(\eta)\mathrm{d}\eta\le\\&\le\frac{1}{1-d}\int\limits_0^{\infty}\boldsymbol{q}^T(\eta)\boldsymbol{X}\boldsymbol{q}(\eta)\mathrm{d}\eta\end{aligned} \tag{35}$$

Conversely, the integral norm-weighting matrix in (35) is independent of h as long as h is strictly greater than 0. Using (35) property then

$$\begin{aligned}&\int\limits_0^{\infty}\left[\,\boldsymbol{q}^T(t-\tau(t))\ \boldsymbol{q}^T(t)\,\right]\begin{bmatrix}\boldsymbol{X}&\\&\boldsymbol{X}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}(t-\tau(t))\\\boldsymbol{q}(t)\end{bmatrix}\mathrm{d}t\le\\&\le\int\limits_0^{\infty}\boldsymbol{q}^T(\eta)\boldsymbol{X}\boldsymbol{q}(\eta)\mathrm{d}\eta+\frac{1}{1-d}\int\limits_0^{\infty}\boldsymbol{q}^T(\eta)\boldsymbol{X}\boldsymbol{q}(\eta)\mathrm{d}\eta=\\&=\frac{2-d}{1-d}\int\limits_0^{\infty}\boldsymbol{q}^T(\eta)\boldsymbol{X}\boldsymbol{q}(\eta)\mathrm{d}\eta\end{aligned} \tag{36}$$

Considering $|\dot{\tau}(t)|\le d$, $1<d\le 2$ it is evident that (36) is negative.

Summarizing, such forms as (36) cannot be generally included into Lyapunov-Krasovskii functional if $1<d\le 2$ since may cause its negative definiteness, and only the standard form of Lyapunov function is proposed to use.

Theorem 1. Autonomous linear time-delay system (1) is stable for $|\dot{\tau}(t)|\le d$, $1<d\le 2$ if there exist symmetric positive definite matrices $\boldsymbol{P}>0$, $\boldsymbol{Q}>0$, $\boldsymbol{P},\boldsymbol{Q}\in I\!R^{n\times n}$, such that

$$\boldsymbol{P}=\boldsymbol{P}^T>0\qquad \boldsymbol{Q}=\boldsymbol{Q}^T>0 \tag{37}$$

$$\begin{bmatrix}\boldsymbol{\Pi}_{11}&h^2(\boldsymbol{A}+\boldsymbol{A}_d)^T\boldsymbol{Q}\boldsymbol{A}_d+\boldsymbol{P}\boldsymbol{A}_d\\ *&h^2\boldsymbol{A}_d^T\boldsymbol{Q}\boldsymbol{A}_d-\boldsymbol{Q}\end{bmatrix}<0 \tag{38}$$

$$\begin{aligned}\boldsymbol{\Pi}_{11}&=(\boldsymbol{A}+\boldsymbol{A}_d)^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{A}_d)+\\&+h^2(\boldsymbol{A}+\boldsymbol{A}_d)^T\boldsymbol{Q}(\boldsymbol{A}+\boldsymbol{A}_d)\end{aligned} \tag{39}$$

Proof. Lyapunov function candidate can be chosen as

$$\begin{aligned}0<v(\boldsymbol{q}(t))&=\boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t)+\\&+\int\limits_0^{t}(h^2\boldsymbol{y}^{\circ T}(r)\boldsymbol{Q}\boldsymbol{y}^{\circ}(r)-\boldsymbol{u}^{\circ T}(r)\boldsymbol{Q}\boldsymbol{u}^{\circ}(r))\mathrm{d}r\end{aligned} \tag{40}$$

where $\boldsymbol{P}=\boldsymbol{P}^T>0$, $\boldsymbol{Q}=\boldsymbol{Q}^T>0$. Evaluating derivative of $v(\boldsymbol{q}(t))$ with respect to t results in

$$\begin{aligned}\dot{v}(\boldsymbol{q}(t))&=-\boldsymbol{u}^{\circ T}(t)\boldsymbol{Q}\boldsymbol{u}^{\circ}(t)+\\&+(\boldsymbol{q}^T(t)\boldsymbol{A}^{\circ T}+\boldsymbol{u}^{\circ T}(t)\boldsymbol{B}^{\circ T})\boldsymbol{P}\boldsymbol{q}(t)+\\&+\boldsymbol{q}^T(t)\boldsymbol{P}(\boldsymbol{A}^{\circ}\boldsymbol{q}(t)+\boldsymbol{B}^{\circ}\boldsymbol{u}^{\circ}(t))+\\&+h^2(\boldsymbol{C}^{\circ}\boldsymbol{q}(t)+\boldsymbol{D}^{\circ}\boldsymbol{u}^{\circ}(t))^T\boldsymbol{Q}(\boldsymbol{C}^{\circ}\boldsymbol{q}(t)+\boldsymbol{D}^{\circ}\boldsymbol{u}^{\circ}(t))<0\end{aligned} \tag{41}$$

Thus, introducing the composite vector $\boldsymbol{q}^{\circ}(t)$ as follows

$$\boldsymbol{q}^{\circ T}(t)=\left[\,\boldsymbol{q}^T(t)\ \boldsymbol{u}^{\circ T}(t)\,\right] \tag{42}$$

it is possible to write the Lyapunov function derivative (42) as follows

$$\dot{v}(\boldsymbol{q}^{\circ}(t))=\boldsymbol{q}^{\circ T}(t)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(t)<0 \tag{43}$$

where

$$\boldsymbol{P}^{\circ}=\begin{bmatrix}\boldsymbol{P}_{11}^{\circ}&h^2\boldsymbol{C}^{\circ T}\boldsymbol{Q}\boldsymbol{D}^{\circ}+\boldsymbol{P}\boldsymbol{B}^{\circ}\\ *&h^2\boldsymbol{D}^{\circ T}\boldsymbol{Q}\boldsymbol{D}^{\circ}-\boldsymbol{Q}\end{bmatrix}<0 \tag{44}$$

$$\boldsymbol{P}_{11}^{\circ}=\boldsymbol{A}^{\circ T}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}^{\circ}+h^2\boldsymbol{C}^{\circ T}\boldsymbol{Q}\boldsymbol{C}^{\circ} \tag{45}$$

Subsequently, inserting (30) then (44), (45) implies (38), (39). This concludes the proof. ■

5. CONTROL LAW PARAMETER DESIGN

Theorem 2. Linear time-delay system (1) is stable for $|\dot{\tau}(t)| \leq d$, $1 < d \leq 2$ with mentioned quadratic performance $\|\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}\|_\infty \leq \gamma$ if there exist symmetric positive definite matrices $\boldsymbol{X} > 0$, $\boldsymbol{Z} > 0$, $\boldsymbol{X}, \boldsymbol{Z} \in I\!R^{n\times n}$, a matrix $\boldsymbol{Y} \in I\!R^{r\times n}$, and a scalar $\gamma > 0$, $\gamma \in I\!R$ such that

$$\boldsymbol{X} = \boldsymbol{X}^T > 0 \qquad \boldsymbol{Z} = \boldsymbol{Z}^T > 0 \tag{46}$$

$$\begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{A}_d\boldsymbol{Z} & \boldsymbol{B} & h\boldsymbol{X}(\boldsymbol{A}+\boldsymbol{A}_d)^T & \boldsymbol{X}\boldsymbol{C}^T \\ * & -\boldsymbol{Z} & \boldsymbol{0} & h\boldsymbol{Z}\boldsymbol{A}_d^T & \boldsymbol{0} \\ * & * & -\gamma\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{Z} & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{47}$$

$$\boldsymbol{\Gamma}_{11} = (\boldsymbol{A}+\boldsymbol{A}_d)\boldsymbol{X} + \boldsymbol{X}(\boldsymbol{A}+\boldsymbol{A}_d)^T - \boldsymbol{B}\boldsymbol{Y} - \boldsymbol{Y}^T\boldsymbol{B}^T \tag{48}$$

Then the control law gain matrix be computed as

$$\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{X}^{-1} \tag{49}$$

Proof. Choosing Lyapunov function candidate as

$$\begin{aligned} 0 < v(\boldsymbol{q}(t)) &= \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \\ &+ \int_0^t (\boldsymbol{y}^T(r)\boldsymbol{y}(r) - \gamma\boldsymbol{u}^T(r)\boldsymbol{u}(r))\mathrm{d}r + \\ &+ \int_0^t (h^2\boldsymbol{y}^{\circ T}(r)\boldsymbol{Q}\boldsymbol{y}^\circ(r) - \boldsymbol{u}^{\circ T}(r)\boldsymbol{Q}\boldsymbol{u}^\circ(r))\mathrm{d}r \end{aligned} \tag{50}$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$, then derivative evaluating of $v(\boldsymbol{q}(t))$ with respect to t gives

$$\begin{aligned} \dot{v}(\boldsymbol{q}(t)) &= \boldsymbol{y}^T(t)\boldsymbol{y}(t) - \gamma\boldsymbol{u}^T(t)\boldsymbol{u}(t) + \\ &+ (\boldsymbol{q}^T(t)\boldsymbol{A}^{\circ T} + \boldsymbol{u}^{\circ T}(t)\boldsymbol{B}^{\circ T})\boldsymbol{P}\boldsymbol{q}(t) + \\ &+ \boldsymbol{q}^T(t)\boldsymbol{P}(\boldsymbol{A}^\circ\boldsymbol{q}(t) + \boldsymbol{B}^\circ\boldsymbol{u}^\circ(t)) + \\ &+ h^2(\boldsymbol{C}^\circ\boldsymbol{q}(t) + \boldsymbol{D}^\circ\boldsymbol{u}^\circ(t))^T\boldsymbol{Q}(\boldsymbol{C}^\circ\boldsymbol{q}(t) + \boldsymbol{D}^\circ\boldsymbol{u}^\circ(t)) - \\ &- \boldsymbol{u}^{\circ T}(t)\boldsymbol{Q}\boldsymbol{u}^\circ(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{u}^T(t)\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{q}(t) < 0 \end{aligned} \tag{51}$$

Introducing the composite vector $\boldsymbol{q}^\bullet(t)$ as follows

$$\boldsymbol{q}^{\bullet T}(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{u}^{\circ T}(t) & \boldsymbol{u}^T(t) \end{bmatrix} \tag{52}$$

the Lyapunov function derivative (52) takes form

$$\dot{v}(\boldsymbol{q}^\bullet(t)) = \boldsymbol{q}^{\bullet T}(t)\boldsymbol{P}^\bullet\boldsymbol{q}^\bullet(t) < 0 \tag{53}$$

where

$$\boldsymbol{P}^\bullet = \begin{bmatrix} \boldsymbol{P}_{11}^\bullet & h^2\boldsymbol{C}^{\circ T}\boldsymbol{Q}\boldsymbol{D}^\circ + \boldsymbol{P}\boldsymbol{B}^\circ & \boldsymbol{B}\boldsymbol{P} \\ * & h^2\boldsymbol{D}^{\circ T}\boldsymbol{Q}\boldsymbol{D}^\circ - \boldsymbol{Q} & \boldsymbol{0} \\ * & * & -\gamma\boldsymbol{I}_r \end{bmatrix} < 0 \tag{54}$$

$$\boldsymbol{P}_{11}^\bullet = \boldsymbol{A}^{\circ T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^\circ + h^2\boldsymbol{C}^{\circ T}\boldsymbol{Q}\boldsymbol{C}^\circ + \boldsymbol{C}^T\boldsymbol{C} \tag{55}$$

Thus, inequality (55) can be written as

$$\boldsymbol{P}^\bullet = \boldsymbol{P}_1^\bullet + \boldsymbol{P}_2^\bullet + \boldsymbol{P}_3^\bullet \tag{56}$$

with

$$\boldsymbol{P}_1^\bullet = \begin{bmatrix} \boldsymbol{A}^{\circ T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^\circ & \boldsymbol{P}\boldsymbol{B}^\circ & \boldsymbol{P}\boldsymbol{B} \\ * & -\boldsymbol{Q} & \boldsymbol{0} \\ * & * & -\gamma\boldsymbol{I}_r \end{bmatrix} \tag{57}$$

$$\begin{aligned} \boldsymbol{P}_2^\bullet &= \begin{bmatrix} h^2\boldsymbol{C}^{\circ T}\boldsymbol{Q}\boldsymbol{C}^\circ & h^2\boldsymbol{C}^{\circ T}\boldsymbol{Q}\boldsymbol{D}^\circ & \boldsymbol{0} \\ h^2\boldsymbol{D}^{\circ T}\boldsymbol{Q}\boldsymbol{C}^\circ & h^2\boldsymbol{D}^{\circ T}\boldsymbol{Q}\boldsymbol{D}^\circ & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \\ &= \begin{bmatrix} \begin{bmatrix} h\boldsymbol{C}^{\circ T} \\ h\boldsymbol{D}^{\circ T} \end{bmatrix} \boldsymbol{Q} \begin{bmatrix} h\boldsymbol{C}^\circ & h\boldsymbol{D}^\circ \end{bmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \geq 0 \end{aligned} \tag{58}$$

$$\boldsymbol{P}_3^\bullet = \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^T \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \geq 0 \tag{59}$$

Now, using Schur complement property it yields

$$\boldsymbol{P}_2^\bullet = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & h\boldsymbol{C}^{\circ T} \\ * & \boldsymbol{0} & \boldsymbol{0} & h\boldsymbol{D}^{\circ T} \\ * & * & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{Q}^{-1} \end{bmatrix} \geq 0 \tag{60}$$

$$\boldsymbol{P}_3^\bullet = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{C}^T \\ * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_m \end{bmatrix} \geq 0 \tag{61}$$

and subsequently

$$\boldsymbol{P}^\bullet = \begin{bmatrix} \boldsymbol{A}^{\circ T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^\circ & \boldsymbol{P}\boldsymbol{B}^\circ & \boldsymbol{P}\boldsymbol{B} & h\boldsymbol{C}^{\circ T} & \boldsymbol{C}^T \\ * & -\boldsymbol{Q} & \boldsymbol{0} & h\boldsymbol{D}^{\circ T} & \boldsymbol{0} \\ * & * & -\gamma\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{Q}^{-1} & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{62}$$

Defining the congruence transform matrix

$$\boldsymbol{T} = \mathrm{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{Q}^{-1} & \boldsymbol{I}_r & \boldsymbol{I} & \boldsymbol{I}_m \end{bmatrix} \tag{63}$$

and pre-multiplying left-hand side as well as right-hand side of (62) by $\boldsymbol{T}$ gives

$$\begin{bmatrix} \boldsymbol{P}_{11}^\diamond & \boldsymbol{B}^\circ\boldsymbol{Q}^{-1} & \boldsymbol{B} & h\boldsymbol{P}^{-1}\boldsymbol{C}^{\circ T} & \boldsymbol{P}^{-1}\boldsymbol{C}^T \\ * & -\boldsymbol{Q}^{-1} & \boldsymbol{0} & h\boldsymbol{Q}^{-1}\boldsymbol{D}^{\circ T} & \boldsymbol{0} \\ * & * & -\gamma\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{Q}^{-1} & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{64}$$

$$\boldsymbol{P}_{11}^\diamond = \boldsymbol{P}^{-1}\boldsymbol{A}^{\circ T} + \boldsymbol{A}^\circ\boldsymbol{P}^{-1} \tag{65}$$

Denoting

$$\boldsymbol{P}^{-1} = \boldsymbol{X},\ \boldsymbol{Q}^{-1} = \boldsymbol{Z},\ \boldsymbol{Y} = \boldsymbol{K}\boldsymbol{P}^{-1} \tag{66}$$

and inserting

$$\begin{aligned} \boldsymbol{A}^\circ\boldsymbol{P}^{-1} &= (\boldsymbol{A}+\boldsymbol{A}_d)\boldsymbol{P}^{-1} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{P}^{-1} = \\ &= (\boldsymbol{A}+\boldsymbol{A}_d)\boldsymbol{X} - \boldsymbol{B}\boldsymbol{Y} \end{aligned} \tag{67}$$

$$\boldsymbol{C}^\circ = \boldsymbol{A}+\boldsymbol{A}_d, \quad \boldsymbol{B}^\circ = \boldsymbol{D}^\circ = \boldsymbol{A}_d \tag{68}$$

then (64), (65) implies (47), (48). This concludes the proof. ■

6. ILLUSTRATIVE EXAMPLE

The system is given by (1), (2), where $h = 2.5$,

$$\boldsymbol{A} = \begin{bmatrix} -2.6 & 0.0 & 0.8 \\ -1.2 & 0.2 & 0.0 \\ 0.0 & 0.5 & -3.0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix}$$

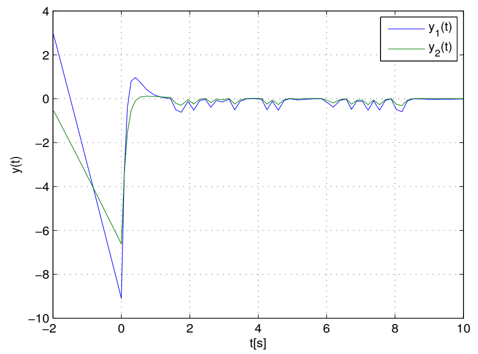

Fig. 1. Output of the system

$$\boldsymbol{A}_d = \begin{bmatrix} 0.00 & 0.02 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ -0.02 & 0.00 & 0.00 \end{bmatrix}, \; \boldsymbol{C}^T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Solving (46), (47) with respect to LMI matrix variables $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, γ using SeDuMi (Self-Dual-Minimization) package for Matlab (Peaucelle et al. (1994)) given task was feasible with

$$\boldsymbol{X} = \begin{bmatrix} 1.7362 & -0.7762 & -0.0051 \\ -0.7762 & 1.0524 & -0.1625 \\ -0.0051 & -0.1625 & 0.8490 \end{bmatrix}$$

$$\boldsymbol{Z} = \begin{bmatrix} 3.4802 & 0.0113 & 0.0097 \\ 0.0113 & 6.4817 & 0.0071 \\ 0.0097 & 0.0071 & 0.5562 \end{bmatrix}$$

$$\boldsymbol{Y} = \begin{bmatrix} 0.0611 & 1.6975 & -0.3312 \\ -0.7696 & 0.6231 & -0.0458 \end{bmatrix}$$

$$\gamma = 5.0659$$

and giving the control system parameters as follows

$$\boldsymbol{K} = \begin{bmatrix} 1.1377 & 2.4656 & 0.0886 \\ -0.2642 & 0.4004 & 0.0210 \end{bmatrix}$$

$$\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{BK} = \begin{bmatrix} -2.0716 & -0.8008 & 0.7579 \\ -4.3488 & -7.5972 & -0.2867 \\ -1.1377 & -1.9656 & -3.0886 \end{bmatrix}$$

$$\boldsymbol{A}_{cs} = \boldsymbol{A} + \boldsymbol{A}_d - \boldsymbol{BK} = \begin{bmatrix} -2.0716 & -0.7808 & 0.7579 \\ -4.3488 & -7.5972 & -1.2867 \\ -1.1577 & -1.9656 & -3.0886 \end{bmatrix}$$

$$\rho(\boldsymbol{A}_c) = \{-1.3917, \; -3.2964, \; -8.0692\}$$

$$\rho(\boldsymbol{A}_{cs}) = \{-1.3742, \; -2.9561, \; -8.4271\}$$

It is evident, that the both sets of eigenvalues spectra $\rho(\boldsymbol{A}_c)$, $\rho(\boldsymbol{A}_{cs})$ of the closed loop system matrices are stable.

In the presented Fig. 1 the example is shown of the unforced closed-loop system output response, where the initial state was $\boldsymbol{q}^T(-2) = [-1 \; 0.5 \; 3]$, $h = 2.5$, $1 < d \leq 2$. It is possible to verify that closed-loop dynamic properties for this unstable autonomous time-delay system are better than any obtained using results implying from Lyapunov-Krasovskii inequality (Lyapunov-Krasovskii functional can stay negative).

7. CONCLUDING REMARKS

Stability conditions for autonomous linear time delay systems as well as the feedback control gain matrix parameter design method are derived in the paper. Considering the delay parameter as an unknown time-varying function with given upper bounds on the magnitude and the variation, the influence of time-varying delay is considered as perturbation in the system, and the presented algorithm gives necessary and sufficient conditions for design in the sense of H_∞ control if the time-delay variation is from the specified range. The advantage of this approach is that the results can be easily generalized for systems with multiple delays, and extended to deal with systems with parametric uncertainties.

ACKNOWLEDGMENTS

The work presented in this paper was supported by VEGA, Grant Agency of Ministry of Education and Academy of Science of Slovak Republic under Grant No. 1/0256/11, as well by Research & Development Operational Programme Grant No. 26220120030 realized in Development of Center of Information and Communication Technologies for Knowledge Systems. These supports are very gratefully acknowledged.

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