Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

of the 18$^{\mathrm{th}}$ International Conference on Process Control
Hotel Titris, Tatranská Lomnica, Slovakia, June 14 – 17, 2011
ISBN 978-80-227-3517-9
http://www.kirp.chtf.stuba.sk/pc11

Editors: M. Fikar and M. Kvasnica

Equivalent Representations of Bounded Real Lemma

D. Krokavec and A. Filasová

Department of Cybernetics and Artificial Intelligence
Faculty of Electrical Engineering and Informatics
Technical University of Košice, Košice, Slovakia
fax: ++ 421 55 625 3574
e-mail: dusan.krokavec@tuke.sk, anna.filasova@tuke.sk

Abstract: The paper concerns the problem of the bounded real lemma for linear continuous-time systems. Using free weighting matrices to express the relationship between the terms of the system state equation a modified equivalent LMI approach to bounded-real-lemma representation is presented. Immediate extension to design method of a memory-free feedback controller, which performs H_∞ properties of the closed-loop system, is formulated as a feasibility problem and expressed over a set of LMIs. Numerical example is included to illustrate the feasibility and properties of the proposed representations.

Keywords: Bounded real lemma, H_∞ performance, continuous-time systems, linear matrix inequality representation.

1. INTRODUCTION

Over the past decade, H_∞ theory seems to be one of the most sophisticated frameworks for robust control system design. Based on concept of quadratic stability which attempts to find a quadratic Lyapunov function (LF), H_∞ norm computation problem is transferred into a standard linear matrix inequality (LMI) optimization task, which includes bounded real lemma (BRL) formulation (Hermann et al. (2007), Veselý and Rosinová (2009), Wu et al. (2010)). A number of more or less conservative analysis methods are presented to assess robust stability for linear systems using a fixed Lyapunov function.

The first version of the BRL presents simple conditions under which a transfer function is contractive on the imaginary axis. Using it, it was possible to determine the H_∞ norm of a transfer function, and the BRL became a significant element to shown and prove that the existence of feedback controllers (that results in a closed loop transfer matrix having the H_∞ norm less than a given upper bound), is equivalent to the existence of solutions of certain LMIs (Boyd *et al.* (1994), Filasová et al. (2010)). Linear matrix inequality approach based on convex optimization algorithms is extensively applied to solve the above mentioned problem (Jia (2003), Pipeleers et al. (2009)) since it can be solved numerically efficiently by using developed interior-point algorithm.

In this paper, equivalent LMI representations of BRL for linear continuous-time systems are introduced. Motivated by the underlying ideas in Filasová and Krokavec (2009), Wu and Duan (2006), and Xie (2008) a simple technique for the BRL representation of linear systems is presented, and used modifications are explained in a context. The proposed LMI representations are proven to be necessary and sufficient and their extensions to state feedback controller design, performing system H_∞ properties is immediate. Translating into LMI framework the closed-loop system stability is characterized in the terms of convex LMIs.

2. PROBLEM DESCRIPTION

Through this paper the task is concerned with the computation of a state feedback $\boldsymbol{u}(t)$, which control the linear dynamic system given by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) + \boldsymbol{D}\boldsymbol{u}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in I\!\!R^n$, $\boldsymbol{u}(t) \in I\!\!R^r$, and $\boldsymbol{y}(t) \in I\!\!R^m$ are vectors of the state, input and measurable output variables, respectively, nominal system matrices $\boldsymbol{A} \in I\!\!R^{n\times n}$, $\boldsymbol{B} \in I\!\!R^{n\times r}$, $\boldsymbol{C} \in I\!\!R^{m\times n}$ and $\boldsymbol{D} \in I\!\!R^{m\times r}$ are real matrices.

Problem of the interest is to design asymptotically stable closed-loop system with the linear memoryless state feedback controller of the form

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}(t) \tag{3}$$

where matrix $\boldsymbol{K} \in I\!\!R^{r\times n}$ is a gain matrix.

3. BASIC PRELIMINARIES

Proposition 1. (Bounded real lemma) System (1), (2) is asymptotically stable if there exist a symmetric positive definite matrix $\boldsymbol{P} > 0$ and a positive scalar $\gamma > 0$ such that

$$\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{4}$$

where $\boldsymbol{I}_r \in I\!\!R^{r\times r}$, $\boldsymbol{I}_m \in I\!\!R^{m\times m}$ are identity matrices, respectively, Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. (see. e.g. Krokavec and Filasová (2008)) Defining Lyapunov function as follows

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \int_0^t (\boldsymbol{y}^T(r)\boldsymbol{y}(r) - \gamma^2\boldsymbol{u}^T(r)\boldsymbol{u}(r))\mathrm{d}r > 0 \tag{5}$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{P} \in I\!R^{n\times n}$, $\gamma > 0 \in I\!R$, and evaluating the derivative of $v(\boldsymbol{q}(t))$ with respect to t along a system trajectory then it yields

$$\begin{aligned}\dot{v}(\boldsymbol{q}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) +\\ +\boldsymbol{y}^T(t)\boldsymbol{y}(t) - \gamma^2\boldsymbol{u}^T(t)\boldsymbol{u}(t) < 0\end{aligned} \tag{6}$$

Thus, substituting (1), (2) into (6) gives

$$\begin{aligned}\dot{v}(\boldsymbol{q}(t)) = (\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t))^T\boldsymbol{P}\boldsymbol{q}(t) +\\ +\boldsymbol{q}^T(t)\boldsymbol{P}(\boldsymbol{A}\boldsymbol{q}(t)+\boldsymbol{B}\boldsymbol{u}(t)) - \gamma\boldsymbol{u}^T(t)\boldsymbol{u}(t) +\\ +(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t))^T(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t)) < 0\end{aligned} \tag{7}$$

and with the next notation

$$\boldsymbol{q}_c^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) \end{bmatrix} \tag{8}$$

it is obtained

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}_c^T(t)\boldsymbol{P}_c\boldsymbol{q}_c(t) < 0 \tag{9}$$

where

$$\boldsymbol{P}_c = \begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B} \\ * & -\gamma^2\boldsymbol{I}_r \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} < 0 \tag{10}$$

Since

$$\begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^T \\ \boldsymbol{D}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \geq 0 \tag{11}$$

Schur complement property implies

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{C}^T \\ * & \boldsymbol{0} & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} \geq 0 \tag{12}$$

and using (12) the LMI condition (10) can be written compactly as (4). This concludes the proof. ■

4. IMPROVED BRL REPRESENTATION

Theorem 1. System (1), (2) is asymptotically stable if there exist a symmetric positive definite matrix $\boldsymbol{P} > 0$, $\boldsymbol{P} \in I\!R^{n\times n}$, matrices $\boldsymbol{S}_1, \boldsymbol{S}_2 \in I\!R^{n\times n}$, and a positive scalar $\gamma > 0$, $\gamma \in I\!R$ such that

$$\begin{bmatrix} -\boldsymbol{S}_1\boldsymbol{A} - \boldsymbol{A}^T\boldsymbol{S}_1^T & -\boldsymbol{S}_1\boldsymbol{B} & \boldsymbol{P}+\boldsymbol{S}_1-\boldsymbol{A}^T\boldsymbol{S}_2^T & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & -\boldsymbol{B}^T\boldsymbol{S}_2^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}_2+\boldsymbol{S}_2^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{13}$$

Proof. Since (1) implies

$$\dot{\boldsymbol{q}}(t) - \boldsymbol{A}\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{u}(t) = \boldsymbol{0} \tag{14}$$

then with arbitrary square matrices $\boldsymbol{S}_1, \boldsymbol{S}_2 \in I\!R^{n\times n}$ it yields

$$\big(\boldsymbol{q}^T(t)\boldsymbol{S}_1 + \dot{\boldsymbol{q}}^T(t)\boldsymbol{S}_2\big)\big(\dot{\boldsymbol{q}}(t) - \boldsymbol{A}\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{u}(t)\big) = 0 \tag{15}$$

Thus, adding (15), as well as its transposition to (6) and substituting (2) it can be written

$$\begin{aligned}\dot{v}(\boldsymbol{q}(t)) = -\gamma\boldsymbol{u}^T(t)\boldsymbol{u}(t) + \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) +\\ +(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t))^T(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t)) +\\ +\big(\dot{\boldsymbol{q}}(t) - \boldsymbol{A}\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{u}(t)\big)^T\big(\boldsymbol{S}_1^T\boldsymbol{q}(t) + \boldsymbol{S}_2^T\dot{\boldsymbol{q}}(t)\big) +\\ +(\boldsymbol{q}^T(t)\boldsymbol{S}_1 + \dot{\boldsymbol{q}}^T(t)\boldsymbol{S}_2)(\dot{\boldsymbol{q}}(t) - \boldsymbol{A}\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{u}(t)) < 0\end{aligned} \tag{16}$$

and using the notation

$$\boldsymbol{q}_c^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) & \dot{\boldsymbol{q}}^T(t) \end{bmatrix} \tag{17}$$

it can be obtained

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}_c^T(t)\boldsymbol{P}_c^\circ\boldsymbol{q}_c(t) < 0 \tag{18}$$

where

$$\begin{aligned}\boldsymbol{P}_c^\circ = \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} & \boldsymbol{0} \\ * & \boldsymbol{D}^T\boldsymbol{D} & \boldsymbol{0} \\ * & * & \boldsymbol{0} \end{bmatrix} +\\ + \begin{bmatrix} -\boldsymbol{S}_1\boldsymbol{A} - \boldsymbol{A}^T\boldsymbol{S}_1^T & -\boldsymbol{S}_1\boldsymbol{B} & \boldsymbol{P}+\boldsymbol{S}_1-\boldsymbol{A}^T\boldsymbol{S}_2^T \\ * & -\gamma^2\boldsymbol{I}_r & -\boldsymbol{B}^T\boldsymbol{S}_2^T \\ * & * & \boldsymbol{S}_2+\boldsymbol{S}_2^T \end{bmatrix} < 0\end{aligned} \tag{19}$$

Thus, analogously using (11), (12) the inequality (19) can be written compactly as (13). This concludes the proof. ■

Remark 1. Setting $\boldsymbol{S}_1 = -\boldsymbol{P}$ then (13) is transformed in

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B} & -\boldsymbol{A}^T\boldsymbol{S}_2^T & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & -\boldsymbol{B}^T\boldsymbol{S}_2^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}_2+\boldsymbol{S}_2^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{20}$$

Thus, inserting $\boldsymbol{S}_2 = -\delta\boldsymbol{I}$, where $\delta > 0$, $\delta \in I\!R$ gives

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B} & \delta\boldsymbol{A}^T & \boldsymbol{C}^T \\ * & -\gamma\boldsymbol{I}_r & \delta\boldsymbol{B}^T & \boldsymbol{D}^T \\ * & * & -2\delta\boldsymbol{I}_n & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B} & \boldsymbol{A}^T & \boldsymbol{C}^T \\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{B}^T & \boldsymbol{D}^T \\ * & * & -2\delta^{-1}\boldsymbol{I}_n & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{22}$$

respectively. Then (22) can be written as

$$\begin{aligned}\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B} & \boldsymbol{C}^T \\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} +\\ +0.5\,\delta \begin{bmatrix} \boldsymbol{A}^T \\ \boldsymbol{B}^T \\ \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} < 0\end{aligned} \tag{23}$$

Choosing δ as a sufficiently small positive scalar satisfying the condition

$$0 < \delta < 2\frac{\lambda_1}{\lambda_2} \tag{24}$$

$$\lambda_1 = \lambda_{min}\left\{-\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix}\right\} \tag{25}$$

$$\lambda_2 = \lambda_{max}\left\{\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{A} & \boldsymbol{A}^T\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{B}^T\boldsymbol{A} & \boldsymbol{B}^T\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\right\} \tag{26}$$

(21) be negative definite for a feasible $\boldsymbol{P}$ of (4).

Corollary 1. Setting $\boldsymbol{S}_1 = -\boldsymbol{P}$, and $\boldsymbol{S}_2 = \delta\boldsymbol{I}$, where $0 < \delta \in I\!R$ then (20)-(22) implies

$$\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{PA} & \boldsymbol{PB} & \boldsymbol{C}^T \\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} - \\ -0.5\,\delta\begin{bmatrix} -\boldsymbol{A}^T \\ -\boldsymbol{B}^T \\ \boldsymbol{0} \end{bmatrix}\begin{bmatrix} -\boldsymbol{A} & -\boldsymbol{B} & \boldsymbol{0} \end{bmatrix} < 0 \tag{27}$$

and a feasible solution $\boldsymbol{P}$ of (4) is also a feasible solution of (27) for all $\delta > 0$, $\delta \in I\!R$.

Theorem 2. System (1), (2) is asymptotically stable if there exist a symmetric positive definite matrix $\boldsymbol{P} > 0$, $\boldsymbol{P} \in I\!R^{n\times n}$, matrices $\boldsymbol{S}_1$, $\boldsymbol{S}_2 \in I\!R^{n\times n}$, and a positive scalar $\gamma > 0$, $\gamma \in I\!R$ such that

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & \boldsymbol{P}+\boldsymbol{S}_1+\boldsymbol{A}^T\boldsymbol{S}_2 & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{B}^T\boldsymbol{S}_2 & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}_2+\boldsymbol{S}_2^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{28}$$

Proof. Defining the congruence transform matrix

$$\boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{I} & & & \\ & \boldsymbol{I} & & \\ \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{I} & \\ & & & \boldsymbol{I} \end{bmatrix} \tag{29}$$

and multiplying right-hand side of (13) by $\boldsymbol{T}_1$ and left-hand side of (13) by $\boldsymbol{T}_1^T$ then after tedious calculation (28) is obtained. This concludes the proof. ■

Remark 2. Setting $\boldsymbol{S}_1 = -\boldsymbol{P}$, $\boldsymbol{S}_2 = -\delta\boldsymbol{P}$ then (28) leads to

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & -\delta\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & -\delta\boldsymbol{B}^T\boldsymbol{P} & \boldsymbol{D}^T \\ * & * & -2\delta\boldsymbol{P} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & -\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & -\boldsymbol{B}^T\boldsymbol{P} & \boldsymbol{D}^T \\ * & * & -2\delta^{-1}\boldsymbol{P} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{31}$$

respectively, and using Schur complement property then (31) can be rewritten as

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} + \\ +\frac{\delta}{2}\begin{bmatrix} -\boldsymbol{A}^T\boldsymbol{P} \\ -\boldsymbol{B}^T\boldsymbol{P} \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{P}^{-1}\begin{bmatrix} -\boldsymbol{PA} & -\boldsymbol{PA} & \boldsymbol{0} \end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} + \\ +\frac{\delta}{2}\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{PA} & \boldsymbol{A}^T\boldsymbol{PB} & \boldsymbol{0} \\ \boldsymbol{B}^T\boldsymbol{PA} & \boldsymbol{B}^T\boldsymbol{PB} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} < 0 \tag{33}$$

respectively. Choosing δ satisfying (24), then with (25) and

$$\lambda_2 = \lambda_{max}\left\{\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{PA} & \boldsymbol{A}^T\boldsymbol{PB} & \boldsymbol{0} \\ \boldsymbol{B}^T\boldsymbol{PA} & \boldsymbol{B}^T\boldsymbol{PB} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\right\} \tag{34}$$

(31) be negative definite for a feasible $\boldsymbol{P}$ of (4). This concludes the proof. ■

Corollary 2. Considering (32), (33) it is evident that the inequality

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & \boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{B}^T\boldsymbol{P} & \boldsymbol{D}^T \\ * & * & -2\delta^{-1}\boldsymbol{P} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{35}$$

and (31) are equivalent.

5. CONTROL LAW PARAMETER DESIGN

Theorem 3. Closed-loop system (1), (2), (3) is stable if there exists a symmetric positive definite matrix $\boldsymbol{X} > 0$, $\boldsymbol{X} \in I\!R^{n\times n}$, a regular square matrix $\boldsymbol{Z} \in I\!R^{n\times n}$, a matrix $\boldsymbol{Y} \in I\!R^{r\times n}$, and a scalar $\gamma > 0$, $\gamma \in I\!R$ such that

$$\boldsymbol{X} = \boldsymbol{X}^T > 0, \quad \gamma > 0 \tag{36}$$

$$\begin{bmatrix} \boldsymbol{\Pi}_{11} & \boldsymbol{B} & \boldsymbol{XA}^T-\boldsymbol{Y}^T\boldsymbol{B}^T & \boldsymbol{XC}^T-\boldsymbol{Y}^T\boldsymbol{D}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{B}^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{Z}+\boldsymbol{Z}^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{37}$$

$$\boldsymbol{\Pi}_{11} = \boldsymbol{AX}+\boldsymbol{XA}^T-\boldsymbol{BY}-\boldsymbol{Y}^T\boldsymbol{B}^T \tag{38}$$

The control law gain matrix is given as

$$\boldsymbol{K} = \boldsymbol{YX}^{-1} \tag{39}$$

Proof. Setting $\boldsymbol{S}_1 = -\boldsymbol{P}$ then (28) implies

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{PB} & \boldsymbol{A}^T\boldsymbol{S}_2 & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{B}^T\boldsymbol{S}_2 & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}_2+\boldsymbol{S}_2^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{40}$$

Supposing that $\det(\boldsymbol{S}_2) \neq 0$ then it can be defined the congruence transform matrix

$$\boldsymbol{T}_2 = \mathrm{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{I}_r & \boldsymbol{S}_2^{-1} & \boldsymbol{I}_m \end{bmatrix} \tag{41}$$

and pre-multiplying right-hand side of (40) by $\boldsymbol{T}_2$, and left-hand side of (40) by $\boldsymbol{T}_2^T$ leads to

$$\begin{bmatrix} \boldsymbol{AP}^{-1}+\boldsymbol{P}^{-1}\boldsymbol{A}^T & \boldsymbol{B} & \boldsymbol{P}^{-1}\boldsymbol{A}^T & \boldsymbol{P}^{-1}\boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{B}^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}_2^{-1}+\boldsymbol{S}_2^{-T} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{42}$$

Thus, denoting

$$\boldsymbol{P}^{-1} = \boldsymbol{X}, \qquad \boldsymbol{S}_2^{-1} = \boldsymbol{Z} \tag{43}$$

(42) can be written as

$$\begin{bmatrix} AX+XA^T & B & XA^T & XC^T \\ * & -\gamma^2 I_r & B^T & D^T \\ * & * & Z+Z^T & 0 \\ * & * & * & -I_m \end{bmatrix} < 0 \tag{44}$$

Inserting $A \leftarrow A_c = A-BK$, and $C \leftarrow C_c = C-DK$ it yields

$$\begin{bmatrix} \Pi_{11} & B & X(A^T-K^TB^T) & X(C^T-K^TD^T) \\ * & -\gamma^2 I_r & B^T & D^T \\ * & * & Z+Z^T & 0 \\ * & * & * & -I_m \end{bmatrix} < 0 \tag{45}$$

where

$$\Pi_{11} = AX+XA^T-BKX-XK^TB^T \tag{46}$$

and with

$$Y = KX \tag{47}$$

(45), (46) implies (37), (38), respectively. This concludes the proof. ■

Remark 3. Setting $Z = -\delta X$ then with $D = 0$ the control law design condition (36)-(38) can be rewritten as

$$X = X^T > 0, \quad \delta > 0 \tag{48}$$

$$\begin{bmatrix} \Pi_{11} & B & XA^T-Y^TB^T & XC^T \\ * & -\gamma^2 I_r & B^T & 0 \\ * & * & -2\delta I_n & 0 \\ * & * & * & -I_m \end{bmatrix} < 0 \tag{49}$$

$$\Pi_{11} = AX+XA^T-BY-Y^TB^T \tag{50}$$

where feasible X, Y, δ implies gain matrix parameter (39).

Therefore, it is evident that the design standard form of BRL is

$$\begin{bmatrix} \Pi_{11} & B & X(C^T-K^TD^T) \\ * & -\gamma^2 I_r & 0 \\ * & * & -I_m \end{bmatrix} < 0 \tag{51}$$

Note, other nontrivial solutions can be obtained using different setting of $S_l,\ l = 1, 2$.

6. ILLUSTRATIVE EXAMPLE

The approaches given above are illustrated by the numerical example where the parameters of (1), (2) are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix},\ B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 5 \end{bmatrix},\ C^T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Solving (48), (49) with respect to LMI matrix variables X, Y, γ, and δ using SeDuMi (Self-Dual-Minimization) package for Matlab (Peaucelle et al. (1994)) given task was feasible with

$$X = \begin{bmatrix} 3.7160 & -2.6784 & 1.2147 \\ -2.6784 & 3.0184 & -1.8970 \\ 1.2147 & -1.8970 & 3.2896 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.8937 & 2.1673 & -1.4078 \\ -0.0801 & -0.0207 & 0.5383 \end{bmatrix}$$

$$\gamma = 11.0242, \qquad \delta = 6.7040$$

and results the control system parameters

$$K = \begin{bmatrix} 2.2731 & 3.0405 & 0.4860 \\ 0.0152 & 0.1662 & 0.2538 \end{bmatrix}$$

$$\rho(A_c) = \{-0.9398,\ -3.1252,\ -11.2561\}$$

It is evident, that the eigenvalues spectrum $\rho(A_c)$ of the closed control loop is stable.

Solving (48), (51) with respect to LMI matrix variables X, Y, and γ given task was feasible, too. Obtained LMI variables were

$$X = \begin{bmatrix} 2.7637 & -1.7983 & 0.6386 \\ -1.7983 & 2.2479 & -1.2081 \\ 0.6386 & -1.2081 & 3.0925 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.9127 & 1.6581 & -0.8163 \\ 0.2802 & 0.1304 & -0.2269 \end{bmatrix}$$

$$\gamma = 6.9412$$

and implies

$$K = \begin{bmatrix} 1.7557 & 2.2852 & 0.2662 \\ 0.2837 & 0.2709 & -0.0261 \end{bmatrix}$$

$$\rho(A_c) = \{-0.8968,\ -5.8435 \pm 1.7282\,\mathrm{i}\}$$

It is evident, that performance γ is less then one obtained with respect to (49) but this fetches worst dynamic properties.

It also should be noted, the cost value γ will not be a monotonously decreasing function with the decreasing of δ, if δ is chosen.

7. CONCLUDING REMARKS

This paper describes a simple technique for equi-valent BRL representation and its application to the H_∞ control of linear systems. Standard criterion is extended for a system with constant coefficient matrices employing free weighting matrices to take the relationship between the terms of the system equation into account in the structure of BRL. The method is further extended to the design of an H_∞ state-feedback controller. Numerical example demonstrates that principles described in this paper are effective, although some computational complexity is increases.

The advantage of this approach is that in Theorem 1 Lyapunov matrix P is separated from A, B^T, C, and D^T, i.e. there are no terms containing the product of P and any of them. This enables a new robust BRL to be derived for a system with polytopic uncertainties by using a parameter-dependent Lyapunov function, and to deal with linear systems with parametric uncertainties. It seems to be a useful extension to other control performance synthesis problems, too.

ACKNOWLEDGMENTS

The work presented in this paper was supported by VEGA, Grant Agency of Ministry of Education and Academy of Science of Slovak Republic under Grant No. 1/0256/11. This support is very gratefully acknowledged.

REFERENCES

D. Boyd, L. El Ghaoui, E. Peron, and V. Bala-krishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM Society for Industrial and Applied Mathematics, Philadelphia, 1994.

A. Filasová and D. Krokavec. Global asymptotically stable control design for time-delay systems. *AT&P Journal Plus*, 2, 89-92, 2009.

A. Filasová, D. Gontkovič, and D. Krokavec. Output feedback control design using unified algebraic approach, In *Proceedings of the 8th International Symposium on Applied Machine Intelligence and Informatics SAMI 2010*, Herľany, Slovakia, 259-262, 2010.

G. Herrmann, M.C. Turner, and I. Postlethwaite. Linear matrix inequalities in control, In *Mathematical Methods for Robust and Nonlinear Control*, Springer–Verlag, Berlin, 123-142, 2007.

Y. Jia. Alternative proofs for improved LMI representations for the analysis and the design of continuous-time systems with poolytopic type uncertainty: A predictive approach, *IEEE Transactions on Automatic Control*, 48:8, 1413-1416, 2003.

D. Krokavec and A. Filasová. *Discrete-Time Systems*, Elfa, Košice, 2008. (in Slovak)

D. Peaucelle, D. Henrion, Y. Labit, and K. Taitz. *User's Guide for SeDuMi Interface 1.04*, LAAS-CNRS, Toulouse, 2002.

G. Pipeleers, B. Demeulenaerea, J. Sweversa, and L. Vandenbergheb. Extended LMI characterizations for stability and performance of linear systems, *Systems & Control Letters*, 58, 510-518, 2009.

V. Veselý and A. Rosinová. Robust output model predictive control design. BMI approach, *International Journal of Innovative Computing, Information and Control*, 5:4, 1115-1123, 2009.

A.I. Wu and G.R. Duan. Enhanced LMI Representations for H_2 Performance of Polytopic Uncertain Systems: Continuous-time Case, *International Journal of Automation and Computing*, 3, 304-308, 2006.

M. Wu, Y. He, and J.H. She. *Stability Analysis and Robust Control of Time-Delay Systems*, Science Press, Beijing and Springer–Verlag, Berlin, 2010.

W. Xie. An equivalent LMI Representation of Bounded Real Lemma for Continous-Time Systems, Journal of Inequalities and Applications, 5, 2010.