Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

of the 18$^{\mathrm{th}}$ International Conference on Process Control

Hotel Titris, Tatranská Lomnica, Slovakia, June 14 – 17, 2011

ISBN 978-80-227-3517-9

http://www.kirp.chtf.stuba.sk/pc11

Editors: M. Fikar and M. Kvasnica

Veselý, V., Rosinová, D.: Robust PSD Controller Design, Editors: Fikar, M., Kvasnica, M., In *Proceedings of the 18th International Conference on Process Control*, Tatranská Lomnica, Slovakia, 565–570, 2011.

Robust PSD controller design

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Abstract: A state space approach to a design of PSD robust controllers is studied for linear uncertain system with affine (polytopic) uncertainty. The discrete time PSD controller design is based on stability condition derived using parameter dependent Lyapunov-Krasovskii function in the form for time-delay system. The resulting design employs solution of BMI, the results are illustrated on the example.

1. INTRODUCTION

Robustness belongs to important issues in control design for real plants. In practice, uncertainties are always present in modelling and control of real systems (modelling errors due to linearization and approximation, disturbances etc.), which must be taken into consideration. The appropriate control has to cope with uncertainties and guarantee closed loop stability and required performance qualities overall the uncertainty domain. Various approaches have been developed in robust control both in time and frequency domains. A frequently used paradigm developed in past decades formulates the problem of robust stability and robust control as an optimization problem. Efficient computational techniques have been developed recently for solving Linear Matrix Inequality (LMI), which enables to solve a large set of convex problems in polynomial time (e.g. Boyd et al., 1994). Significant effort has been made in this field to formulate control problems within algebraic framework (Skelton et al., 1998), and transform them into LMI. Though many control problems for uncertain linear systems can be formulated as convex one, there are still many important control problems even for linear systems, that have been proven as NP hard (Blondel and Tsitsiklis, 1997). Robust static output feedback (SOF) control belongs also to this class, generally it can be formulated as bilinear matrix inequality (BMI). In this case either solution through BMI solver (as PENBMI) can be used, or, convex approximation or linearization can be applied (deOliveira et al., 2000; Han and Skelton, 2003, Veselý, 2003; Rosinová and Veselý, 2003). Characterization of basic LMI and BMI features in control problems can be found in (Van Antwerp and Braatz 2000).

Proportional-integral-derivative (PID) controllers belong to the most popular and frequent ones in the industrial applications. For a discrete time case often PSD abbreviation is used instead of PID, where “S” stands for a summation term (instead of integration). Results on LMI approach to design PID controller can be found e.g. in (Ge et al. 2002, Zheng Feng et al. 2000). Robust PSD controller design can be treated as dynamic controller, which further can be formulated as SOF problem for augmented system including controller dynamics, (Rosinová and Veselý, 2007). As indicated above, SOF problem, which is generally non-convex, can be either solved by solving BMI or by linearization (convex approximation) and then through LMI.

In this paper, another approach is proposed for robust PSD controller design, where the difference term is considered in the framework of time-delay systems and the respective Lyapunov-Krasovskii function comprising two parts. In Section 2, robust PSD controller problem is formulated for linear uncertain polytopic system with quadratic performance index. Section 3 presents the main result- robust stability condition with guaranteed cost formulated for robust PSD controller. This condition is developed using parameter dependent Lyapunov function in the form for time- delay systems. The proposed approach is illustrated on the example in Section 4.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the class of linear uncertain discrete-time systems described as:

$$\begin{aligned} x(t+1) &= A(\alpha)x(t) + B(\alpha)u(t) \\ y(t) &= Cx(t) \end{aligned} \qquad t = 0,1,... \tag{1}$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^l$ are state, control and output vectors respectively; uncertain model matrices $A(\alpha), B(\alpha)$ are from convex polytopic uncertainty domain given by polytope vertices $A_j \in R^{nxn}, B_j \in R^{nxm}$, $j = 1,...,N$:

$$(A(\alpha), B(\alpha)) \in S,$$
$$S = \left\{ (A(\alpha), B(\alpha)) \colon\ A(\alpha) = \sum_{j=1}^{N} \alpha_j A_j, B(\alpha) = \sum_{j=1}^{N} \alpha_j B_j, \sum_{j=1}^{N} \alpha_j = 1,\ \alpha_j \geq 0 \right\} \tag{2}$$

2.1 PSD controller

Assuming that the input reference variable $w(t)$ is constant (its changes are relatively slow in comparison with system dynamics), PSD control algorithm for uncertain system (1), (2) can be considered as

$$u(t) = k_p y(t) + k_i \sum_{i=0}^{t} y_i(k) + k_d [y_d(t) - y_d(t-1)] \tag{3}$$

where $y_i(t) = C_i x(t), y_d(t) = C_d x(t)$ denote the respective outputs for summation (discrete approximation of integration) and difference term of control algorithm respectively, which in general can differ from output $y(t)$; k_p, k_i, k_d are constant matrices of corresponding dimensions.

In parallel with standard approach for continuous-time case, to include summation term we introduce augmented state vector defined by

$$z(t) = \begin{bmatrix} x(t) \\ r(t) \end{bmatrix}$$

where $r(t+1) = r(t) + y_i(t)$, $t = 0,1,...$ (4)

Using (4), PSD control algorithm can be rewritten as

$$\begin{aligned} u(t) &= k_p Cx(t) + k_i r(t) + k_d C_d x(t) - k_d C_d x(t-1) = \\ &= (k_p C + k_d C_d) x(t) + k_i r(t) - k_d C_d x(t-1) \\ &= \begin{bmatrix} k_p C + k_d C_d & k_i \end{bmatrix} z(t) - k_d C_d x(t-1) \end{aligned} \tag{5}$$

From (1), (4) and (5) we obtain the uncertain closed-loop polytopic system described in a compact form as

$$\begin{aligned} z(t+1) &= A_C(\alpha) z(t) + B_C(\alpha) K_C C_C z(t) - B_C(\alpha) k_d C_{d1} z(t-1) = \\ &= (A_C(\alpha) + B_C(\alpha) K_C C_C) z(t) - B_C(\alpha) k_d C_{d1} z(t-1) \end{aligned} \tag{6}$$

where

$$A_C(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ C_i & I \end{bmatrix},\ B_C(\alpha) = \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix},\ K_C = \begin{bmatrix} k_p & k_d & k_i \end{bmatrix}$$

$$C_C = \begin{bmatrix} C & 0 \\ C_d & 0 \\ 0 & I \end{bmatrix},\ C_{d1} = [C_d \ \ 0] \tag{7}$$

2.2 Performance index

A performance for closed loop system (6) is assessed using quadratic cost function

$$J = \sum_{t=0}^{\infty} J(t)$$

$$J(t) = z(t)^T Q z(t) + u(t)^T R u(t) + z(t-1)^T S\, z(t-1) \tag{8}$$

where $Q, S \in R^{n \times n}, R \in R^{m \times m}$ are symmetric positive definite matrices.

Definition 1

Control law (5) is called guaranteed cost control when there exist PID controller parameter matrices k_p, k_i, k_d and a constant J_0 such that

$$J \le J_0$$

holds for closed loop system (6); J_0 is the guaranteed cost.

2.3 Robust stability with guaranteed cost

Let $V(t)$ is Lyapunov function for uncertain closed-loop system (6). From LQ theory, see e.g. (Rosinova et al. 2003), the following lemma for robust stability of system (6) with guaranteed cost holds.

Lemma 1

Control algorithm (5) is the guaranteed cost control law for the closed loop system (6) if and only if there exist $V(t) > 0$ and constant matrices k_p, k_i, k_d such that the following inequality holds for $t = 0,1,...$

$$B(t) = \Delta V(t) + J(t) < 0 \tag{9}$$

Moreover, summarizing (9) from initial time t_0 to $t \to \infty$, the following inequality is obtained

$$-V(t_0) + J < 0 \tag{10}$$

Definition 1 with inequality (10) provides guaranteed cost

$$J_0 = V(t_0)$$

for closed loop system (6) with control law (5).

3. MAIN RESULT: ROBUST PSD CONTROLLER DESIGN

In this section a robust stability condition including guaranteed cost is developed based on Lyapunov-Krasovskii function. Due to the presence of $z(t-1)$ in control algorithm, we consider parameter dependent Lyapunov-Krasovskii function consisting of the respective two parts for $z(t)$ and $z(t-1)$

$$V(t) = V_1(t) + V_2(t) \tag{11}$$

where $V_1(t) = z(t)^T P_1(\alpha) z(t)$ (11a)

$V_2(t) = z(t-1)^T P_2(\alpha) z(t-1)$ (11b)

and $P_1(\alpha) \in R^{n \times n}$, $P_2(\alpha) \in R^{n \times n}$ are parameter dependent symmetric positive definite matrices of the corresponding dimensions.

In the following developments we employ the backward difference given by

$$\Delta z(t) = z(t) - z(t-1) \tag{12}$$

"to interconnect" actual and past values of z, which can be interpreted as discrete counterpart to Leibnitz-Newton formula used for continuous time-delay systems.

In the development of a robust stability condition for uncertain closed-loop system we use augmented state vector

$$v(t) = \begin{bmatrix} z(t+1) \\ z(t) \\ \Delta z(t) \end{bmatrix}. \tag{13}$$

Firstly, we express particular components which will be used later in terms of denotation (7), (12) and (13).

Control law (5) can be rewritten as

$$\begin{aligned} u(t) &= K_C C_C\, z(t) - k_d C_d x(t-1) \\ &= K_C C_C\, z(t) - k_d C_{d1}(z(t) - \Delta z(t)) \\ &= \begin{bmatrix} 0 & K_C C_C - k_d C_{d1} & k_d C_{d1} \end{bmatrix} v(t) \end{aligned} \tag{14}$$

Closed loop system (6) can be analogically rewritten as

$$z(t+1) = \widetilde{A}_C(\alpha) z(t) + B_C(\alpha) k_d C_{d1} \Delta z(t) \tag{15}$$

with

$$\widetilde{A}_C(\alpha) = A_C(\alpha) + B_C(\alpha) K_C C_C - B_C(\alpha) k_d C_{d1}$$

from which we have

$$\begin{bmatrix} I & -\widetilde{A}_C(\alpha) & -B_C(\alpha) k_d C_{d1} \end{bmatrix} \begin{bmatrix} z(t+1) \\ z(t) \\ \Delta z(t) \end{bmatrix} = 0 \tag{16}$$

The first difference of Lyapunov-Krasovskii function (11) is

$$\Delta V(t) = \Delta V_1(t) + \Delta V_2(t) \tag{17a}$$

$$\Delta V_1(t) = z(t+1)^T P_1(\alpha) z(t+1) - z(t)^T P_1(\alpha) z(t) \tag{17b}$$

$$\begin{aligned} \Delta V_2(t) &= z(t)^T P_2(\alpha) z(t) - z(t-1)^T P_2(\alpha) z(t-1) = \\ &= -\Delta z(t)^T P_2(\alpha) \Delta z(t) + z(t)^T P_2(\alpha) \Delta z(t) + \Delta z(t)^T P_2(\alpha) z(t) \end{aligned} \tag{17c}$$

Substituting (17b) and (17c) into (17a) and rearranging, we obtain $\Delta V(t)$ in a compact form as

$$\Delta V(t) = v(t)^T \begin{bmatrix} P_1(\alpha) & 0 & 0 \\ 0 & -P_1(\alpha) & P_2(\alpha) \\ 0 & P_2(\alpha) & -P_2(\alpha) \end{bmatrix} v(t) \tag{18}$$

The main result on robust stability condition is given in the next theorem.

Theorem 1

Consider the uncertain discrete-time system (1) with PID controller (3) and parameter dependent Lyapunov-Krasovskii function (11),(11a),(11b). Control algorithm (3), or, alternatively, (14) is guaranteed cost robust control law for performance index (8) if and only if there exist positive definite matrices $P_1(\alpha) \in R^{n \times n}$, $P_2(\alpha) \in R^{n \times n}$ and constant matrices N_1, N_2, N_3 of appropriate dimensions such that

$$v(t)^T W(\alpha) v(t) < 0 \tag{19}$$

where $v(t)$ is given in (13) and $W(\alpha) = \{w_{ij}(\alpha)\}$:

$$\begin{aligned} w_{11}(\alpha) &= N_1 + N_1^T + P_1(\alpha) \\ w_{12}(\alpha) &= -N_1^T \widetilde{A}_C(\alpha) + N_2 \\ w_{13}(\alpha) &= -N_1^T B_C(\alpha) k_d C_{d1} + N_3 \\ w_{22}(\alpha) &= -N_2^T \widetilde{A}_C(\alpha) - \widetilde{A}_C^T(\alpha) N_2 - P_1(\alpha) + \\ &+ (K_C C_C - k_d C_{d1})^T R (K_C C_C - k_d C_{d1}) + S + Q \\ w_{23}(\alpha) &= -N_2^T B_C(\alpha) k_d C_{d1} - \widetilde{A}_C^T(\alpha) N_3 + P_2(\alpha) + \\ &+ (K_C C_C - k_d C_{d1})^T R k_d C_{d1} - S \\ w_{33}(\alpha) &= -N_3^T B_C(\alpha) k_d C_{d1} - C_{d1}^T k_d^T B_C^T(\alpha) N_3 - P_2(\alpha) + S + \\ &+ C_{d1}^T k_d{}^T R k_d C_{d1} \end{aligned}$$

Proof

To derive robust stability condition we use (9) together with (16). Due to (16), the following equality holds

$$\begin{aligned} & v(t)^T \begin{bmatrix} N_1^T \\ N_2^T \\ N_3^T \end{bmatrix} \begin{bmatrix} I & -\widetilde{A}_C(\alpha) & -B_C(\alpha) k_d C_{d1} \end{bmatrix} v(t) + \\ & + v(t)^T \begin{bmatrix} I \\ -\widetilde{A}_C^T \\ -C_{d1}^T k_d^T B_C^T(\alpha) \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} v(t) = 0 \end{aligned} \tag{20}$$

By substituting from (17), (14) and (8) to (9) for $J(t)$ and $\Delta V(t)$ respectively, and adding (20) to the left hand side of (9), the resulting inequality (19) is obtained. □

Robust stability condition (19) can be advantageously applied for polytopic systems with uncertainties respective to (2). Parameter dependent Lyapunov-Krasovskii function is considered in the form

$$\begin{aligned} P_1(\alpha) &= \sum_{i=1}^{N} \alpha_i P_{1i} \quad \text{where } P_{1i} = P_{1i}^T > 0 \\ P_2(\alpha) &= \sum_{i=1}^{N} \alpha_i P_{2i} \quad \text{where } P_{2i} = P_{2i}^T > 0 \end{aligned} \tag{21}$$

Robust stability condition (19) is in this case linear with respect to α (there are no products of matrices depending on α), therefore it is equivalent to

$$v(t)^T W(\alpha_i) v(t) < 0, \; i = 1,..., N \tag{22}$$

where $\sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0$.

Note, that robust stability condition (22) is in LMI form for stability analysis – for unknown matrices $N_1, N_2, N_3, P_{1i}, P_{2i}$. For robust PID controller design, where unknown controller parameter matrices k_p, k_i, k_d are to be found, inequality (22) turns to bilinear matrix inequality (BMI), which can be solved either directly using some BMI solver or through linearization of nonlinear terms, see e.g. (deOliveira et al.

2000). We have applied the former approach and solved BMI (22) via PENBMI solver with YALMIP interface.

4. EXAMPLE

In this section the proposed approach to design a robust PSD controller is illustrated on the example.

Consider uncertain system (1), (2) with 3 states, 2 inputs and 2 outputs with nominal model

$$A_0 = \begin{bmatrix} 0.6 & 0.0097 & 0.0143 \\ 0.012 & 0.945 & 0.0049 \\ -0.0047 & 0.01 & 0.46 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.0425 & 0.0053 \\ 0.0052 & 0.01 \\ 0.0024 & 0.0474 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and uncertainty matrices

$$A_u = \begin{bmatrix} 0.004 & 0 & 0 \\ 0 & 0.055 & 0 \\ 0 & 0.0005 & 0.0002 \end{bmatrix}, B_u = 10^{-4}\begin{bmatrix} 0.2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

The respective uncertain polytopic model vertices for (2) are:

vertex 1: $A_1 = A_0 + A_u, B_1 = B_0 + B_u$,

vertex 2: $A_2 = A_0 - A_u, B_2 = B_0 - B_u$

eigenvalues of vertex system matrices A_1, A_2 are:

$$eig(A_1): \begin{matrix} 1.0004 \\ 0.6032 \\ 0.4606 \end{matrix} \qquad eig(A_2): \begin{matrix} 0.8905 \\ 0.5951 \\ 0.4602 \end{matrix}$$

Vertex 1 corresponds to unstable system.

Performance index (8) is considered, with weighting matrices

$R = I_{2x2}, Q = 0.1 * I_{5x5}, S = 0.1 * I_{5x5}$.

The aim is to find PSD controller parameter matrices k_p, k_i, k_d (of dimensions 2x2) so that the closed loop system (6) is robustly stable with guaranteed cost. The outputs for summation and difference part of control algorithm are $C_i = C$, $C_d = C$.

PSD controller has been designed by solving (19) as BMI. Two alternative PSD controllers have been computed:

using parameter dependent Lyapunov function (11) denoted as PQS

using simple quadratic Lyapunov function, with the same Lyapunov function matrices P_1, P_2 in (11) for the whole uncertainty domain; this case is computed for comparison and denoted as QS

The obtained results are summarized in Tab.1

	PQS	QS
k_p	$\begin{bmatrix} -5.6450 & -0.2486 \\ -0.6602 & -5.257 \end{bmatrix}$	$\begin{bmatrix} -3.7366 & -0.2483 \\ -0.5043 & -3.2184 \end{bmatrix}$
k_i	$\begin{bmatrix} -2.1584 & -0.1348 \\ -0.4630 & -2.4361 \end{bmatrix}$	$\begin{bmatrix} -1.6671 & -0.1716 \\ -0.3897 & -1.8401 \end{bmatrix}$
k_d	$\begin{bmatrix} 1.2917 & -0.1006 \\ 0.0473 & 1.2494 \end{bmatrix}$	$\begin{bmatrix} 0.1044 & -0.0085 \\ 0.0174 & 0.0731 \end{bmatrix}$
Closed-loop system eigenvalues		
Vertex1	0.4054 0.5990 0.7277 0.8415 0.9968	0.4721 0.6962 0.7126 0.8715 0.9968
Vertex2	0.4052 0.5788 0.7412 0.8433 **0.8846**	0.4713 0.6537 0.7500 0.8729 0.8837

Tab.1 PSD controller design results for parameter dependent Lyapunov function (PQS) and quadratic Lyapunov function (QS).

Step responses obtained from simulation in Simulink for vertices A_1 and A_2 are shown in Fig. 1 and 2.

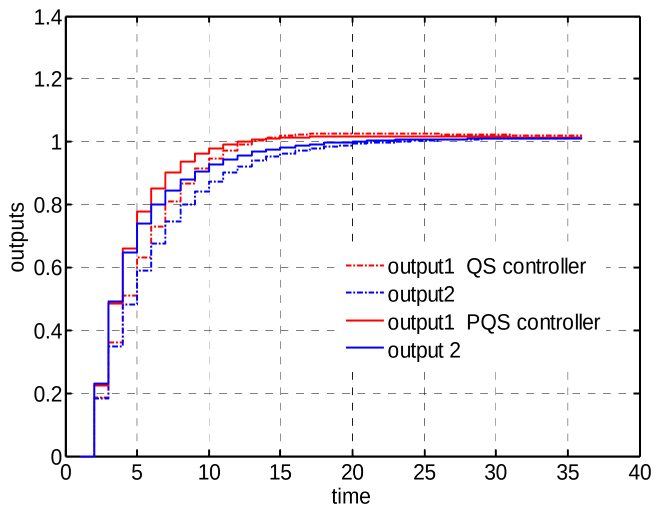

Fig.1 Comparison of closed-loop step responses for PSD controllers QS and PQS in vertex A_1, B_1.

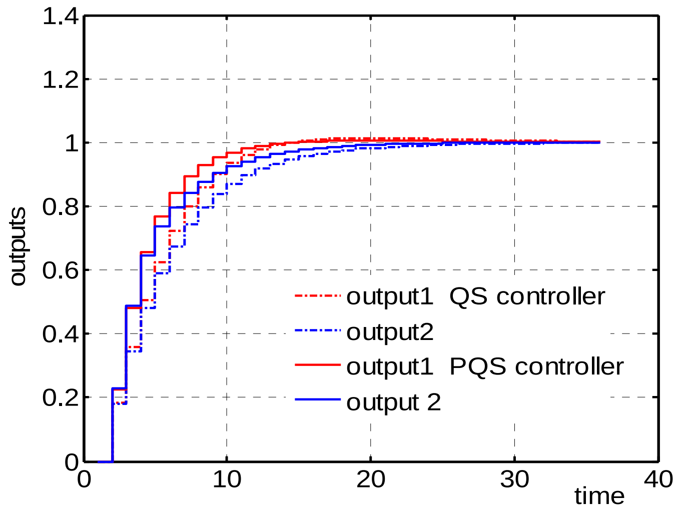

Fig.2 Comparison of closed-loop step responses for PSD controllers QS and PQS in vertex A_2, B_2.

From Fig.1 and 2 it can be seen that the closed-loop dynamics favours parameter dependent Lyapunov function based design over the quadratic one.

The respective control inputs are shown in Fig.3.

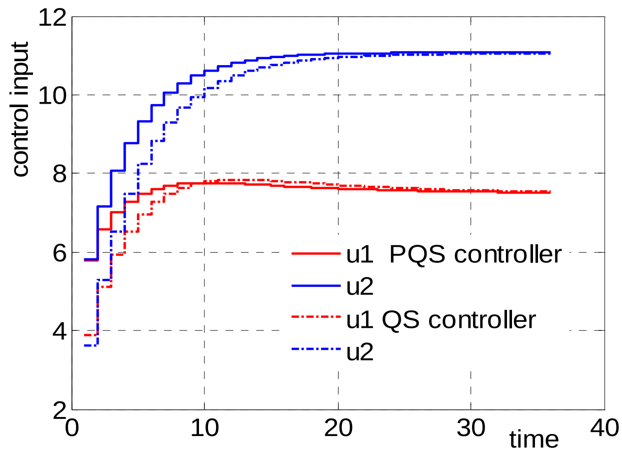

Fig.3 Comparison of control inputs in vertex A_2, B_2

5. CONCLUSION

In the paper the novel PSD controller design procedure is presented, which is based on Lyapunov function with special term corresponding to time-delay part of control algorithm. The resulting robust stability condition is in BMI form, in the illustrating example the PSD controller design has been performed using BMI solver.

ACKNOWLEDGMENTS

The work has been supported by the Slovak Scientific Grant Agency, Grant No. 1/0544/09 and 1/0592/10. This support is very gratefully acknowledged.

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