NTNU
Norwegian University of
Science and Technology

Sum of squares programming for discrete-time non-linear systems - with application to power converters

Morten Hovd and Mohsen Vatani

STUBA, January 27, 2016

Table of Contents

NTNU
Norwegian University of
Science and Technology

Table of Contents

NTNU
Norwegian University of Science and Technology

Linear Matrix Inequalities

- Have found widespread use in control
- Very flexible and practical for specifying conditions on quadratic functions (in control often related to quadratic Lyapunov functions).
- E.g., estimating the rate of decrease of a quadratic Lyapunov function. We want to maximize γ subject to $\frac{d}{dt}(x^T P x) < -\gamma x^T I x$ for the autonomous dynamics $\dot{x} = Ax$. This can be expressed as

$$A^T P + PA < -\gamma I$$

- The $<$ here refers to sign definiteness of matrices. I.e., $(\gamma I + A^T P + PA)$ is required to be a negative definite matrix. The condition on the decrease of the LF is hence fulfilled for any x.
- LMIs constitute *convex* constraints in optimization problems. If we have a convex objective function, depending on decision variables that enter linearly in the LMI constraints, the overall problem is convex (and hence suitable for efficient optimization).

NTNU
Norwegian University of
Science and Technology

Working with Linear Matrix Inequalities

— There is a number of 'standard tricks' to get an expression into the form of an LMI. Will here only mention the Schur complement:

$$E - F^T P^{-1} F > 0 \qquad \text{and} \qquad P > 0 \Longleftrightarrow \begin{bmatrix} E & F^T \\ F & P \end{bmatrix} > 0$$

This follows from the identity

$$\begin{bmatrix} E & F^T \\ F & P \end{bmatrix} = \begin{bmatrix} I_E & F^T P^{-1} \\ 0 & I_P \end{bmatrix} \begin{bmatrix} E - F^T P^{-1} F & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I_E & 0 \\ P^{-1} F & I_P \end{bmatrix}$$

— The Schur complement allows replacing a non-linear expression with a linear expression in a higher-dimensional space.

Using the Schur complement

Controller design for a discrete time system.

$$P - (A+BK)^T P(A+BK) > 0 \Leftrightarrow \begin{bmatrix} P & (A+BK)^T P \\ P(A+BK) & P \end{bmatrix} > 0$$

Pre- and postmultiply with

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} P^{-1} & P^{-1}(A+BK)^T \\ (A+BK)P^{-1} & P^{-1} \end{bmatrix} > 0$$

which is linear in P^{-1} and $Y = KP^{-1}$.

Note the implicit assumption that P^{-1} (and hence also P) is symmetric and positive definite.

NTNU
Norwegian University of
Science and Technology

Non-negative polynomials and sums of squares

— Proving that a polynomial is positive is in general not a tractable problem.

— Most convenient *practical* approach is to prove that the polynomial is a sum of squares

  - Not all positive polynomials are sums of squares

— A sum of squares is non-negative (may be zero if all squared terms share a common root).

— Finding a sum of squares decomposition is a convex problem.

LMIs and SOS

- In LMIs we (implicitly) pre- and post-multiply the matrix with a vector. I.e., $L > 0 \Leftrightarrow x^T L x > 0 \; \forall x \neq 0$.
- SOS problems also use an 'LMI-like' approach. In this case, the vector (implicitly) pre- and postmultiplying the matrix is a vector of *monomials*, e.g. $\begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 & \cdots \end{bmatrix}^T$.
- Since the elements of the vector are not independent, we get extra parameters entering the matrix, for example (from Parillo)

$$F(x_1,x_2) = \quad 2x_1^4 + 2x_1^3 x2 - x_1^2 x_2^2 + 5x_2^4 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -\lambda & 1 \\ -\lambda & 5 & 0 \\ 1 & 0 & -1+2\lambda \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}$$

If a value for λ is found for which the matrix is positive definite, then $F(x_1,x_2)$ is a sum of squares (which is the case in the above example).

NTNU
Norwegian University of Science and Technology

SOS and scalarization

- We may have matrix valued SOS. The polynomial matrix $M(x)$ is a matrix-valued SOS if it can be decomposed as $M(x) = F^T(x)F(x)$.
- For LMIs, using scalar expression has no purpose, $L > 0 \Leftrightarrow x^T L x > 0 \; \forall x \neq 0$.
- For SOS, $M(x) > 0$ is not equivalent to $x^T M(x) x > 0$. $M(x) > 0 \Leftrightarrow z^T M(x) z > 0 \; \forall z$, where there is no relationship between x and z.
- It is therefore much less conservative to express a condition in the scalar form $x^T M(x) x > 0$, if possible.

NTNU
Norwegian University of
Science and Technology

A scalarized Schur complement

Given a matrix

$$M(x) = \begin{bmatrix} E(x) & F^T(x) \\ F(x) & P(x) \end{bmatrix}$$

with $P(x)$ symmetric and invertible. Then

$$\begin{bmatrix} x \\ z \end{bmatrix}^T M(x) \begin{bmatrix} x \\ z \end{bmatrix} > 0, \quad \forall (x,z) \neq (0,0)$$

is equivalent to

$$x^T(E(x) - F^T(x)P(x)^{-1}F(x))x > 0 \forall x \neq 0 \text{ and } z^T P(x) z > 0 \forall z \neq 0$$

NTNU
Norwegian University of
Science and Technology

Scalarized Schur...

This follows from the identity

$$M(x) = \begin{bmatrix} I_E & F^T(x)P^{-1} \\ 0 & I_P \end{bmatrix} \begin{bmatrix} E(x) - F^T(x)P^{-1}F(x) & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I_E & 0 \\ P^{-1}F(x) & I_P \end{bmatrix}$$

by observing that

$$\begin{bmatrix} I_E & 0 \\ P^{-1}F(x) & I_P \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix}$$

Whatever the value of x, w can be chosen to produce any value of z, and *vice versa*.

NTNU
Norwegian University of Science and Technology

Solving LMI and SOS problems

- LMI and SOS problems where decision variables enter linearly, with a convex objective function, are convex optimization problems.
- Decent software exists for these problems. The user can therefore focus on problem formulation, little need to know the details of the mathematical solution procedure.
- Recommended literature for LMIs: Carsten Scherer's course notes. For SOS, some reference literature is also becoming available, e.g., G. Chesi's book on Springer LNCIS.
- ...
- Unfortunately, many problems (especially SOS) result in a bilinear problem formulation. Optimization solvers for bilinear problems exist - but can fail for many problems.
- Iterative solution procedures are therefore often necessary. These can be highly dependent on a good initial point.

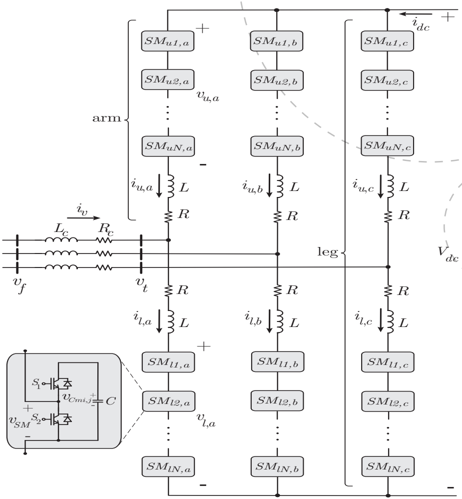
NTNU
Norwegian University of Science and Technology

Table of Contents

NTNU
Norwegian University of Science and Technology

Introduction of MMC

- a large number of voltage cells connected in series
- by inserting desired number of cells, 'any' voltage level can be produced
- less harmonics
- no need for AC filters
- redundancy is higher
- lower switching frequency and semiconductor loss
- reduced manufacturing cost due to similarity of cells



NTNU
Norwegian University of Science and Technology

MMC in HVDC system

- Besides other applications, MMC has become the most promising converter topology for HVDC stations
- MMC-HVDC projects:

Project	Trans Bay	Nanhui	Southwest	Dalian	France	Zhoushan
DC Voltage	± 200 kV	±30 kV	±300 kV	±320 kV	±320 kV	±200 kV
Power	400 MW	20 MW	1440 MW	1000 MW	1000 MW	400 MW
Length	80 km	8.4 km	250 km	43 km	65 km	134 km
operated by	Siemens	C-EPRI	Alstom	C-EPRI	Siemens	C-EPRI
Year	2010	2011	2015	2013	2015	2015
Location	San Francisco	Shanghai	Sweden	China	France	China
type	underwater	offshore windfarm	city connection	under ground	under ground	multi terminal

NTNU
Norwegian University of Science and Technology

MMC in HVDC system

— Tennet off-shore wind farm complex

— in North Sea near to the German coast :

Wind Park	Power (MW)	Voltage (kV)	Cable length (km)	Commissioned by	State
Helwin 1	576	+/- 250	130	Siemens	Started operation in 2013
Dolwin 1	800	+/- 640	165	ABB	Tested during 2013
Borwin 2	800	+/- 300	200	Siemens	Tested during 2013
Sylwin 1	864	+/- 320	205	Siemens	Started operation in 2014
Dolwin 2	900	+/- 640	135	ABB	Started operation in 2015
Dolwin 3	900	+/- 320	162	Alstom	Started operation in 2017

MMC Control difficulties

— the control of the MMC converter is not as easy as other types of converters:

- Control of power transfer
- Balancing capacitor voltages
- Reducing circulating current
- Decrease switching frequency and loss
- Decrease communication load

— The control problem becomes a Multi-Input Multi-Output problem and classical PI controllers does not satisfy the objectives

— Advanced control methods is introduced in recent years for MMC control:

- Repetitive control
- Model Predictive Control
- Proportional Resonant controller
- Optimization with Lagrange multipliers
- ...

NTNU
Norwegian University of Science and Technology

Table of Contents

The goal

- The MMC is modeled as a discrete-time bilinear system

$$x_{k+1} = Ax_k + \sum_{i=1}^{m}(B_i x_k + b_i)u_{i,k} = Ax_k + (B_x + B)u_k$$

- the nonlinearity consists of products between the states and input
- To stabilize the system, the Lyapunov inequality

$$V(x_k) - V(x_{k+1}) = x_k^T P x_k - x_{k+1}^T P x_{k+1} > 0$$

  should be fulfilled.
- The SOS method is used to design the controller, using the YALMIP package (in MATLAB). The controller is in the form of ratio of two polynomials (for each input):

$$u_i(x) = \frac{c_i(x_k)}{c_0(x_k)}$$

NTNU
Norwegian University of
Science and Technology

Controller design by SOS method in YALMIP

Theorem

Region of convergence: Given a quadratic function $V(x) = x^T Px$*, polynomials* $c_i(x), i \in [1,\ldots,m]$*, and SOS polynomials* $\acute{c_0}(x)$ *and* $s_1(x,z)$*, a bilinear discrete time system in closed loop with the control law*

$$u(x) = \frac{C(x)x}{(\acute{c_0}(x)+1)}$$

is stable $\forall x | x^T Px < \gamma$*, provided*

$$\begin{bmatrix} x \\ z \end{bmatrix}^T M(x) \begin{bmatrix} x \\ z \end{bmatrix} - s_1(x,z)(\gamma - x^T Px) > 0$$

where

$$M(x) = \begin{bmatrix} (\acute{c_0}(x)+1)P & ((\acute{c_0}(x)+1)A + (B_x + B)C(x))^T P \\ P((\acute{c_0}(x)+1)A + (B_x + B)C(x)) & (\acute{c_0}(x)+1)P \end{bmatrix}$$

NTNU
Norwegian University of
Science and Technology

Input constraints

Input constraints are easily included in the controller design by another SOS condition (in a way very similar to what is done for LMIs)

NTNU
Norwegian University of Science and Technology

List of variables

— states:

- $\mathbf{i}_{v,dq}$ ac-side currents in *dq* reference frame
- $\mathbf{i}_{cir,dq}$ circulating currents in *dq* reference frame
- i_{d0} dc component of the circulating current
- W the total stored energy in the converter
- ΔW energy difference between the upper and lower arms

— inputs

- $\mathbf{V}_{u,dq}$ upper arm voltage in the *dq* reference frame
- $\mathbf{V}_{l,dq}$ lower arm voltage in the *dq* reference frame
- V_{d0} the dc component of arm voltages

— Other parameters

- ω rotating frequency of source voltage
- $\mathbf{v}_{f,dq}$ the ac-side voltage of the converter

MMC model in the dq frame

— for the ac side current in *dq* reference frame:

$$\frac{d\mathbf{i}_{v,dq}}{dt} = \begin{bmatrix} -\frac{R+2R_c}{L+2L_c} & \omega \\ -\omega & -\frac{R+2R_c}{L+2L_c} \end{bmatrix} \mathbf{i}_{v,dq} + \frac{\mathbf{v}_{u,dq} - \mathbf{v}_{l,dq}}{L+2L_c} + \frac{2\mathbf{v}_{f,dq}}{L+2L_c}.$$

— Circulating current: $\frac{d\mathbf{i}_{\text{cir},dq}}{dt} = \begin{bmatrix} -\frac{R}{L} & \omega \\ -\omega & -\frac{R}{L} \end{bmatrix} \mathbf{i}_{\text{cir},dq} - \frac{1}{2L}(\mathbf{v}_{u,dq} + \mathbf{v}_{l,dq}),$

— dc component of circulating current: $\frac{di_{d0}}{dt} = -\frac{R}{L} i_{d0} - \frac{1}{2L} V_{d0} + \frac{1}{2L} V_{dc}$

— the stored energy dynamics:

$$\frac{dW}{dt} = \frac{dW_u}{dt} + \frac{dW_l}{dt} = -\frac{3}{4} v_{u,d} i_{v,d} + \frac{3}{2} v_{u,d} i_{\text{cir},d} - \frac{3}{4} v_{u,q} i_{v,q} + \frac{3}{2} v_{u,q} i_{\text{cir},q} + \frac{3}{4} v_{l,d} i_{v,d} + \frac{3}{2} v_{l,d} i_{\text{cir},d} + \frac{3}{4} v_{l,q} i_{v,q} + \frac{3}{2} v_{l,q} i_{\text{cir},q} + 3 V_{d0} i_{cir,0},$$

$$\frac{d\Delta W}{dt} = \frac{dW_u}{dt} - \frac{dW_l}{dt} = -\frac{3}{4} v_{u,d} i_{v,d} + \frac{3}{2} v_{u,d} i_{\text{cir},d} - \frac{3}{4} v_{u,q} i_{v,q} + \frac{3}{2} v_{u,q} i_{\text{cir},q} - \frac{3}{4} v_{l,d} i_{v,d} - \frac{3}{2} v_{l,d} i_{\text{cir},d} - \frac{3}{4} v_{l,q} i_{v,q} - \frac{3}{2} v_{l,q} i_{\text{cir},q}.$$

NTNU
Norwegian University of
Science and Technology

MMC model in the dq frame

the bilinear model of MMC

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} -\frac{R+2R_c}{L+2L_c} & w & 0 & 0 & 0 & 0 & 0 \\ -w & -\frac{R+2R_c}{L+2L_c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{R}{L} & w & 0 & 0 & 0 \\ 0 & 0 & -w & -\frac{R}{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{R}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_c} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 \end{bmatrix}}_{B_{1c}} \mathbf{x}(t) u_1$$

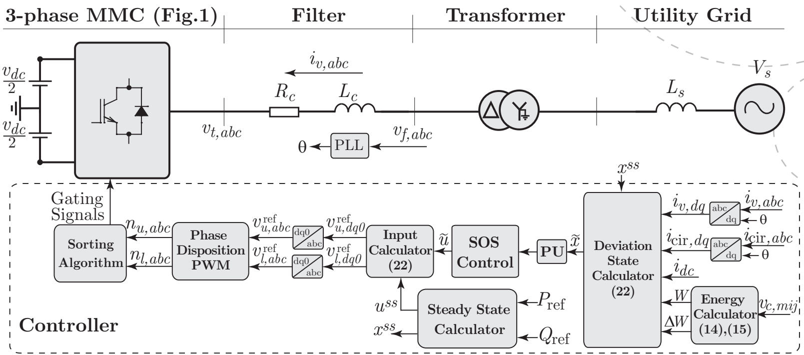
$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 & \frac{3}{2} & 0 & 0 & 0 \end{bmatrix}}_{B_{2c}} \mathbf{x}(t) u_2 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 \end{bmatrix}}_{B_{3c}} \mathbf{x}(t) u_3 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{3}{4} & 0 & \frac{3}{2} & 0 & 0 & \cdots \\ 0 & -\frac{3}{4} & 0 & -\frac{3}{2} & 0 & 0 & \cdots \end{bmatrix}}_{B_{4c}}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{B_{5c}} \mathbf{x}(t) u_5 + \underbrace{\begin{bmatrix} \frac{1}{L+2L_c} \\ 0 \\ -\frac{1}{2L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_{1c}} u_1 + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L+2L_c} \\ 0 \\ -\frac{1}{2L} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_{2c}} u_2 + \underbrace{\begin{bmatrix} -\frac{1}{L+2L_c} \\ 0 \\ -\frac{1}{2L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_{3c}} u_3 + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{L+2L_c} \\ 0 \\ -\frac{1}{2L} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_{4c}} u_4 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2L} \\ 0 \\ 0 \end{bmatrix}}_{b_{4c}} u_5 + \underbrace{\begin{bmatrix} \frac{2v_{f,d}}{L+2L_c} \\ \frac{2v_{f,q}}{L+2L_c} \\ 0 \\ 0 \\ \frac{V_{dc}}{2L} \\ 0 \\ 0 \end{bmatrix}}_{d_c}$$

NTNU
Norwegian University of Science and Technology

Control block diagram

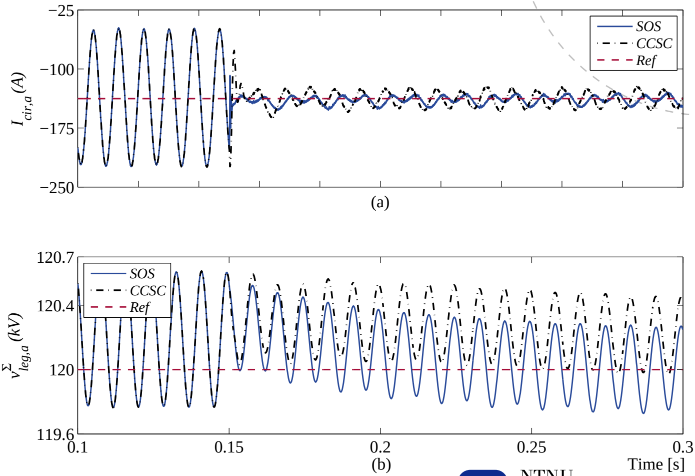
— The following system is simulated in PLECS/MATLAB



NTNU
Norwegian University of Science and Technology

Activation of controller

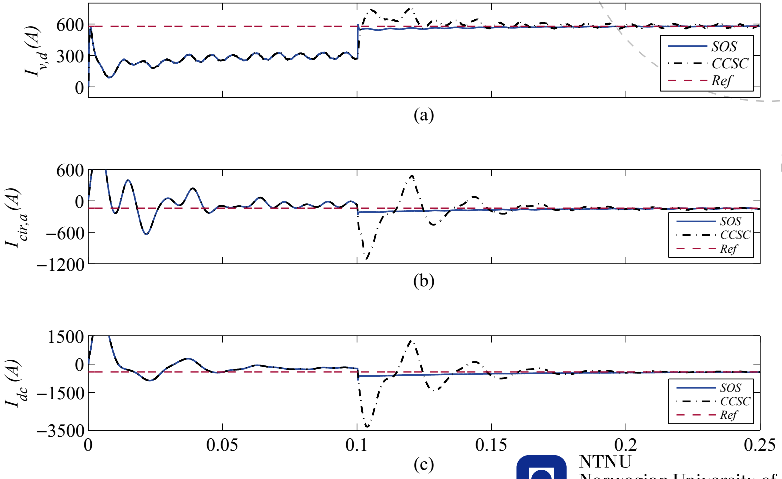
— Activation of controller at $t = 0.15$ s



NTNU
Norwegian University of
Science and Technology

Convergence to the operating point

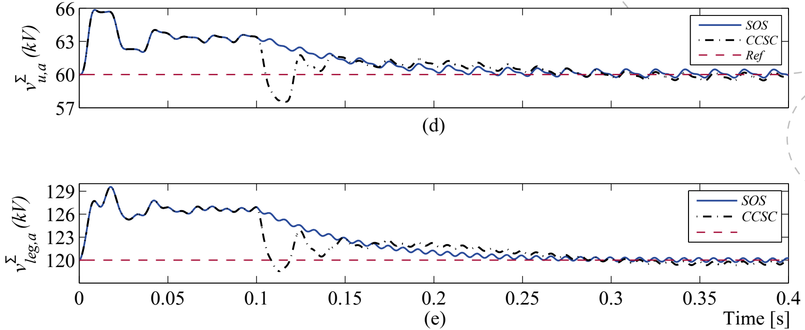
— Before activation of the controller, the states are far from their references.



NTNU
Norwegian University of
Science and Technology

Convergence to the operating point

— Before activation of the controller, the states are far from their references.



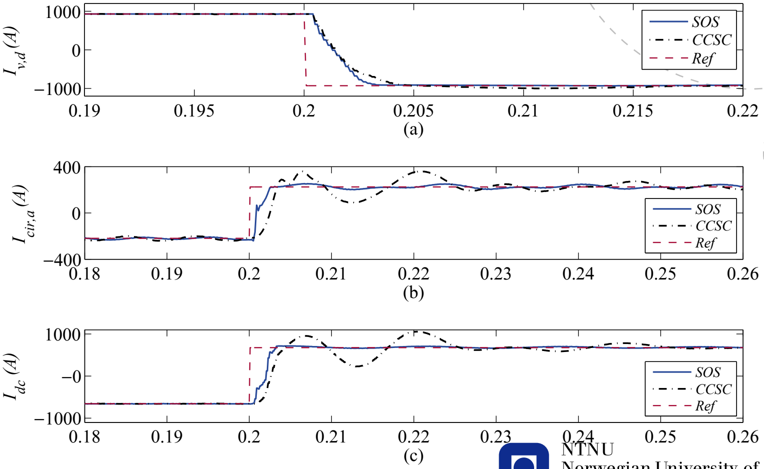
(d)

(e)

NTNU
Norwegian University of
Science and Technology

Real power flow reversal command

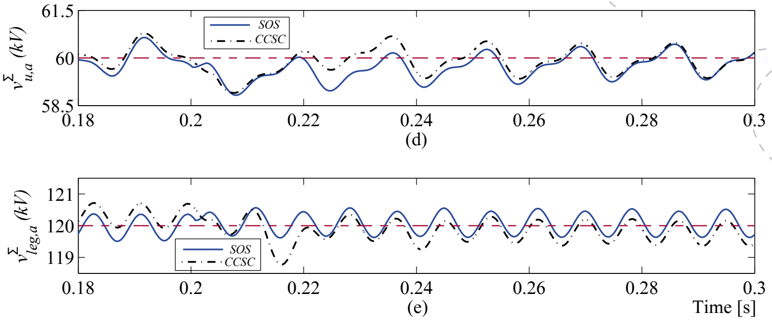
— Initially, the MMC system is in a steady-state condition, transferring $P = 40\,\text{MW}$ to the ac grid. At $t = 0.2\,\text{s}$, the real power flow is reversed to $P = -40\,\text{MW}$.



NTNU
Norwegian University of Science and Technology

Real power flow reversal command

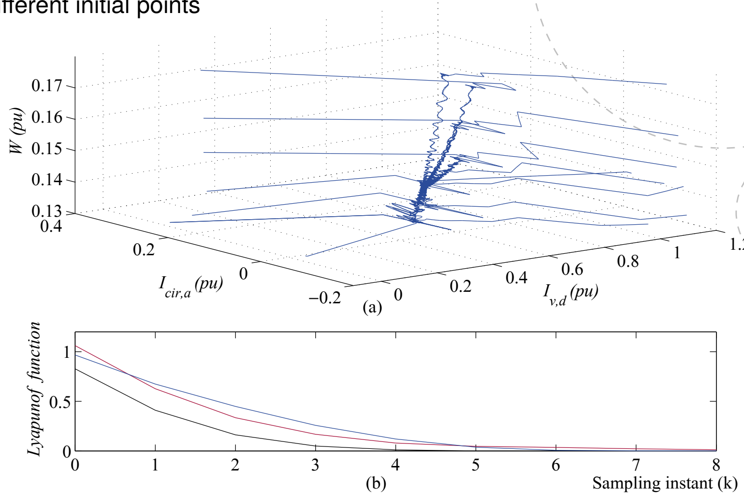
— Initially, the MMC system is in a steady-state condition, transferring $P = 40\,\text{MW}$ to the ac grid. At $t = 0.2\,\text{s}$, the real power flow is reversed to $P = -40\,\text{MW}$.



NTNU
Norwegian University of
Science and Technology

State Trajectories and Lyapunov Function

— for different initial points



NTNU
Norwegian University of
Science and Technology

Table of Contents

NTNU
Norwegian University of
Science and Technology

Power converter modeling

A wide variety of power converters are modeled as switched systems with a specific model for each switching status as

$$\text{switch status } on: \quad \dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{B}_1\mathbf{v},$$
$$\text{switch status } off: \quad \dot{\mathbf{x}}(t) = \mathbf{A}_2\mathbf{x}(t) + \mathbf{B}_2\mathbf{v},$$

where state **x** represents capacitor voltages and inductor currents and vector **v** represents source voltages and diode voltages.

A Pulse Width Modulation (PWM) signal with switching frequency $f_s = 1/T_s$ controls the on/off status of the converter switches. The sum of t_{on} and t_{off} for each switch is equal to the switching period T_s.

The duty cycle d is defined as the ratio of t_{on}/T_s and consequently $t_{off} = (1-d)T_s$.

The duty cycle d is the control input.

NTNU
Norwegian University of Science and Technology

Power converter modeling

Assuming that the inductor current is not saturated and by considering the duty cycle definition, the average model of the converter is formulated as

$$\dot{\mathbf{x}}(t) = (d(t)\mathbf{A}_1 + (1 - d(t))\mathbf{A}_2)\mathbf{x}(t) + (d(t)\mathbf{B}_1 + (1 - d(t))\mathbf{B}_2)\mathbf{v}$$

which can be simplified and reformulated as

$$\dot{\mathbf{x}}(t) = \underbrace{\mathbf{A}_2}_{\mathbf{A}_c}\mathbf{x}(t) + \underbrace{(\mathbf{A}_1 - \mathbf{A}_2)}_{\mathbf{B}_{cb}}\mathbf{x}(t)d(t) + \underbrace{(\mathbf{B}_1\mathbf{v} - \mathbf{B}_2\mathbf{v})}_{\mathbf{B}_c}d(t) + \underbrace{\mathbf{B}_2\mathbf{v}}_{\mathbf{d}_c}. \quad (1)$$

Equation (1) is in the form of a standard bilinear continuous time system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_{cb}\mathbf{x}(t)u(t) + \mathbf{B}_c u(t) + \mathbf{d}_c,$$

where $u(t) = d(t)$ is the input of the system. In general, for converters with more than one switch, the averaged continuous time bilinear model of the converter is represented by

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \sum_{i=1}^{m}(\mathbf{B}_{cb,i}\mathbf{x}(t) + \mathbf{B}_{c,i})\mathbf{u}_i(t) + \mathbf{d}_c, \quad (2)$$

where $u_i = d_i$ is the duty cycle of the ith switch and m is the number of switches.

Introducing deviation variables

The desired equilibrium operating point of the bilinear model of the converter is nonzero. By defining the desired equilibrium state vector and input as $\mathbf{x}^{ss}$ and $\mathbf{d}^{ss}$, respectively, the coordinate transformation is defined by

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^{ss} \quad , \quad \tilde{\mathbf{d}} = \mathbf{d} - \mathbf{d}^{ss}. \tag{3}$$

Substituting for the state variables and input from (3) in (2) yields

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}_c(\tilde{\mathbf{x}}(t) + \mathbf{x}^{ss}) + \sum_{i=1}^{m}(\mathbf{B}_{cb,i}(\tilde{\mathbf{x}}(t) + \mathbf{x}^{ss}) + \mathbf{B}_{c,i})(\tilde{d}_i(t) + d_i^{ss}) + \mathbf{d}_c. \tag{4}$$

Equation (4) can be decomposed into two equations. The first equation represents the desired steady state operation

$$\mathbf{A}_c\mathbf{x}^{ss} + \sum_{i=1}^{m}(\mathbf{B}_{cb,i}\mathbf{x}^{ss} + \mathbf{B}_{c,i})d_i^{ss} + \mathbf{d}_c = 0, \tag{5}$$

while the second equation represents the dynamics of the converter in deviation variables

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}_c + \sum_{i=1}^{m}\mathbf{B}_{cb,i}d_i^{ss})\tilde{\mathbf{x}}(t) + \sum_{i=1}^{m}(\mathbf{B}_{cb,i}\tilde{\mathbf{x}}(t) + \mathbf{B}_{cb,i}\mathbf{x}^{ss} + \mathbf{B}_{c,i})\tilde{d}_i(t). \tag{6}$$

Discrete time model

Based on (6) and assuming a sampling period of T_s, the discrete-time bilinear model of the converter, based on a forward Euler approximation, becomes

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= \underbrace{\left(T_s\mathbf{A}_c + T_s\sum_{i=1}^{m}\mathbf{B}_{cb,i}d_i^{ss} + \mathbf{I}\right)}_{\mathbf{A}}\tilde{\mathbf{x}}_k \\ &+ \sum_{i=1}^{m}(\underbrace{T_s\mathbf{B}_{cb,i}}_{\mathbf{B}_{b,i}}\tilde{\mathbf{x}}_k + \underbrace{T_s\mathbf{B}_{cb,i}\mathbf{x}^{ss} + T_s\mathbf{B}_{c,i}}_{\mathbf{B}_i})\tilde{d}_{i,k}.\end{aligned}$$

which is in the form of a standard discrete-time bilinear system as

$$\tilde{\mathbf{x}}_{k+1} = \mathbf{A}\tilde{\mathbf{x}}_k + \sum_{i=1}^{m}(\mathbf{B}_{b,i}\tilde{\mathbf{x}}_k + \mathbf{B}_i)\tilde{u}_{i,k}. \tag{7}$$

Control of dc-dc boost converter

— Circuit diagram of the boost converter

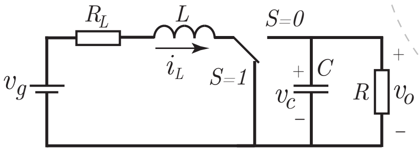


Figure: Circuit diagram of boost converter.

— Model with switch *on*

$$\mathbf{A}_1 = \begin{bmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad v = v_g,$$

— Model with switch off

$$\mathbf{A}_2 = \begin{bmatrix} -\frac{R_L}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad v = v_g.$$

— The load voltage $v_o = v_C = x_2$ is considered as the output which should be kept at the desired voltage $x_2^{ss} = v_{ref} = 24\,V$.

Controller design for dc-dc boost converter

We want to stabilize the discrete-time bilinear average model of the boost converter, in the region determined by $\tilde{\mathbf{x}}^T \mathbf{P} \tilde{\mathbf{x}} < \gamma$. The matrix **P** is selected as:

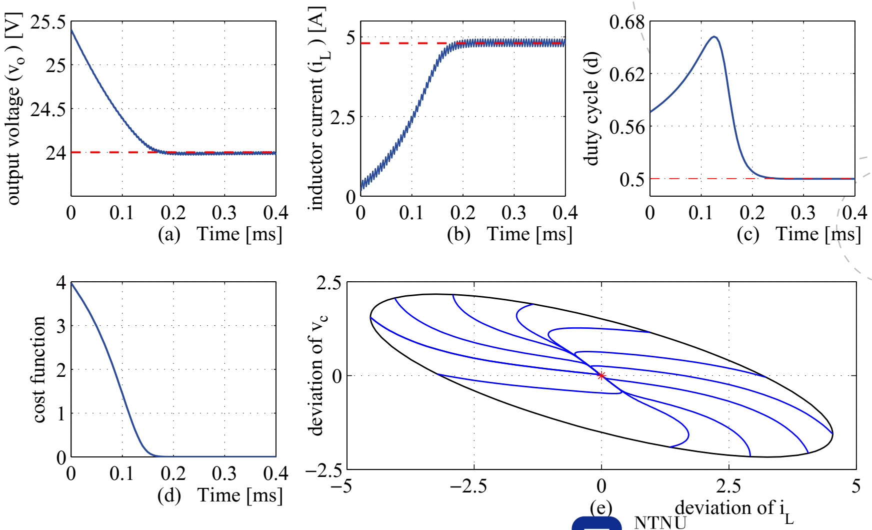
$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 1.75 \end{bmatrix},$$

and $\gamma = 4$. The control effort constraint is set to $\tilde{u}_{max} = 0.4$.

Solving the SOS problem, the following controller is obtained:

$$\tilde{u}(\tilde{\mathbf{x}}) = \frac{-9.1\tilde{x}_1 - 7.21\tilde{x}_2 - 0.14\tilde{x}_1^2 - 0.003\tilde{x}_1\tilde{x}_2 + 0.16\tilde{x}_2^2}{38.29 - 0.46\tilde{x}_1 + 0.23\tilde{x}_2 + 14.71\tilde{x}_1^2 - 2.13\tilde{x}_1\tilde{x}_2 + 11.56\tilde{x}_2^2}.$$

Simulation results



NTNU
Norwegian University of
Science and Technology

Change of operating point

- For linear systems, the dynamics (in terms of deviation variables) do not change when the operating point is changed.
- For nonlinear systems this is not the case. The dynamics (also in terms of deviation variables) will change with change of operating point.
- Assuming that the new operating point is consistent with the steady state equations, how can we guarantee stability of the controlled system around the new operating point?

The dynamics at the new operating point can be expressed as

$$\hat{\mathbf{x}}_{k+1} = (\mathbf{A} + \Delta\mathbf{A})\hat{\mathbf{x}}_k + (\mathbf{B}_{\hat{\mathbf{x}}_k} + \mathbf{B} + \Delta\mathbf{B})\hat{\mathbf{u}}_k. \tag{8}$$

where $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ depend linearly on the change in operating point.

NTNU
Norwegian University of Science and Technology

Change of operating point

- The linear dependence of the dynamics on the change of operating point allows analyzing stability in a way similar to what is done for the control of LPV systems.
- It is also important that the SOS design criterion depends linearly on the system dynamics.
- We cover the range of parameter variations by a polytope.
- If the controller stabilizes the system at all vertices of the polytope, stabilization is also guaranteed for all possible parameter values (operating points) in the interior of the polytope.
- This result follows from a simple interpolation argument - the SOS stability condition at an internal point of the polytope can be found by interpolating between the conditions at the vertices of the polytope.

NTNU
Norwegian University of Science and Technology

DC-DC boost converter revisited

- The same DC-DC boost converter as before.
- Controller design for $x_2^{ss} = v_{ref} = 24V$
- Want to investigate stability of operating points in the range $20V \leq v_{ref} \leq 30V$.
- Steady state operating points and outer-bounding polytope shown in figure.

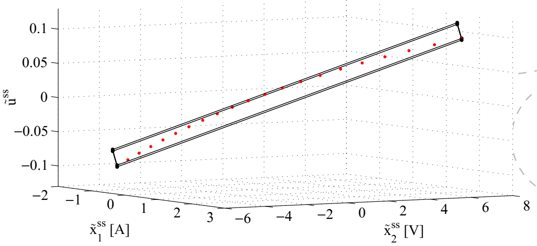


Figure: The set of new operating points and the outer approximated polytope which covers the points

NTNU
Norwegian University of Science and Technology

Introducing integral action

— Common requirement that DC-DC converters should maintain their output voltage close to the reference also in the presence of persistent disturbances.

— The converter load may change, some parameters may drift.

— Integral action required to counteract such effects.

— Integral action introduced in the design my augmenting the model with an integrating state

$$\begin{bmatrix} \tilde{\mathbf{x}}_O \\ \tilde{\mathbf{x}}_I \\ \mathbf{x}_{II} \end{bmatrix}_{k+1} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} & \mathbf{I} \end{bmatrix}}_{\mathbf{A}_I} \begin{bmatrix} \tilde{\mathbf{x}}_O \\ \tilde{\mathbf{x}}_I \\ \mathbf{x}_{II} \end{bmatrix}_k + \Big(\underbrace{\begin{bmatrix} \mathbf{B}_{\tilde{\mathbf{x}}} \\ \mathbf{0} \end{bmatrix}}_{\mathbf{B}_{\tilde{\mathbf{x}},I}} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}_{\mathbf{B}_I} \Big) \tilde{\mathbf{u}}_k. \tag{9}$$

— Here $\mathbf{A}$, $\mathbf{B}_{\tilde{\mathbf{x}}}$, $\mathbf{B}$ describe the dynamics of the original bilinear system, $\mathbf{x}_I$ is the state for which steady state accuracy (integral action) is desired, and $\mathbf{x}_{II}$ is the augmented state.

— The augmented system is still bilinear.

NTNU
Norwegian University of
Science and Technology

DC-DC boost converter with integral action

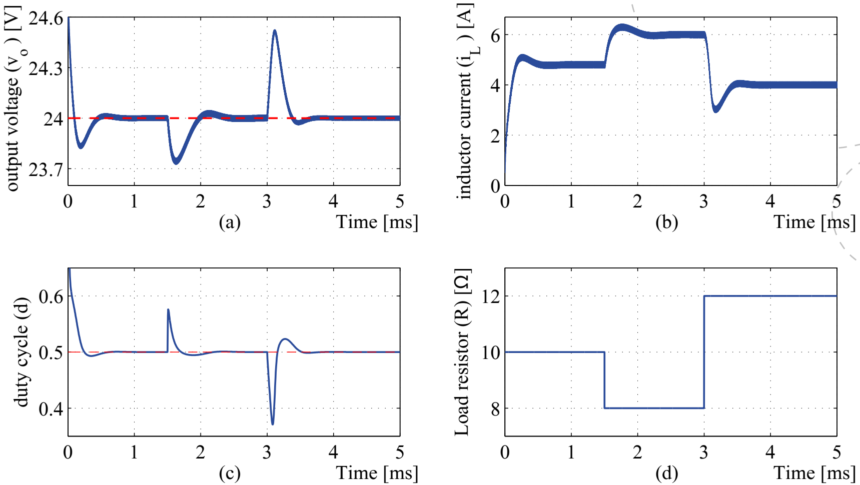


Figure: Simulation results of the boost converter for SOS controller with integral action: (a) output voltage, (b) inductor current, (c) duty cycle of the switch, and (d) load resistor

NTNU
Norwegian University of
Science and Technology

Table of Contents

Conclusions

- Relatively user friendly software makes SOS controller design available to users without detailed knowledge of the optimization solution procedure
  - YALMIP
  - SOSTOOLS
- SOS formulations provides a systematic way of accounting for the bilinear problem structure - and to utilize it in the design.
  - A significant amount of work on more general polynomial nonlinearities by other researchers.
  - Most work in continuous time. Discrete time impose other requirements on the controller structure.
- Practical requirements on the controller design can be handled in ways that closely resemble how this is done for linear systems.
- Although the controller design might be said to be quite mathematically complex, the online controller calculations are quite trivial:
  - Evaluate $n_u + 1$ polynomials
  - Perform n_u scalar divisions

NTNU
Norwegian University of Science and Technology

Outlook

- The number of decision variables in the optimization formulation quickly becomes large, in particular for a large number of states and high orders of the controller polynomials.
- Interesting on-going work (by other people) on imposing/utilizing problem structure in order to handle larger problems.
- SOS formulations also allow for higher order Lyapunov functions
  - The number of decision variables in the design increases quickly
  - Not clear how this is systematically utilized in the design. Some systems require higher-order LFs - but how do we measure the size of the stable region?
- Better methods (incl. improved solvers) for problems where design parameters enter bilinearly would be very welcome.