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SENSITIVITY ANALYSIS OF PARABOLIC SYSTEMS WITH BOUNDARY CONDITIONS INVOLVING TIME DELAYS

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Abstract: In the paper the first order sensitivity analysis is performed for a class of optimal control problems for parabolic equations with the Neumann boundary conditions involving time delays. A singular perturbation of geometrical domain of integration is introduced in the form of a circular hole. The Steklov-Poincaré operator on a circle is defined in order to reduce the problem to regular perturbations in the truncated domain. The optimality system is differentiated with respect to the small parameter and the directional derivative of the optimal control is obtained as a solution to an auxiliary optimal control problem.

Keywords: Shape optimization, topological derivative, optimal design, time delay, parabolic equation.

1. INTRODUCTION

We consider an optimal control problem in the domain with small geometrical defect. The size of the defect is measured by small parameter $\rho > 0$. The presence of the defect results in the singular perturbation of the parabolic state equation. Such a perturbation is transformed to the regular perturbation in the truncated domain Ω_R for any $R > \rho > 0$. We perform the sensitivity analysis in the truncated domain using the Steklov-Poincaré operator defined on the circle Γ_R. The problems of the sensitivity analysis for regular perturbations of optimal control problems were studied in Lasiecka and Sokołowski (1991), Malanowski and Sokołowski (1986), Malanowski (2001), Rao and Sokołowski (2000), Sokołowski (1985), Sokołowski (1987), Sokołowski (1988), Sokołowski and Zolesio (1992). Singular perturbations of geometrical domains are analysed in Jackowska et al. (2002), Jackowska et al. (2003), Maz'ya et al. (2000), Nazarov (1999), Nazarov and Sokołowski (2003a), Nazarov and Sokołowski (2004), Nazarov and Sokołowski (2003c), Nazarov and Sokołowski (2003b), Nazarov et al. (2004), Sokołowski and Żochowski (1999a), Sokołowski and Żochowski (1999b), Sokołowski and Żochowski (1999c), Sokołowski and Żochowski (2001), Sokołowski and Żochowski (2003). The construction of asymptotic approximation for the Steklov-Poincaré operator

is given in Sokołowski and Żochowski (2005).
In the present paper an optimal control problem in singularly perturbed geometrical domain Ω_ρ is analysed with respect to small parameter $\rho > 0$. We derive the one-term asymptotic expansion of optimal controls. The first term of the expansion, of the order ρ^2 is uniquely determined as an optimal solution to the auxiliary optimal control problem. The control constraints for the auxiliary problem are obtained by an application of the conical differentiability of metric projection in L^2 spaces. Our method is constructive and can lead to numerical procedures for determination of the first order approximations of the optimal controls.

2. PRELIMINARIES

Consider now the distributed parameter system described by the following time delay parabolic equation

$$\left.\begin{array}{lll} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in} & \Omega_\rho \times (0,T) \\ \dfrac{\partial y}{\partial \eta} = y(x,t-h) + v & \text{on} & \Gamma \times (0,T) \\ \dfrac{\partial y}{\partial \eta} = 0 & \text{on} & \Gamma_\rho \times (0,T) \\ y(x,0) = y_0(x) & \text{in} & \Omega_\rho \\ y(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0) \end{array}\right\} \quad (1)$$

where:

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

$\partial/\partial\eta$ is a normal derivative at $\Gamma\rho$ directed towards the exterior of $\Omega\rho$, $\Omega\rho$ is presented on the Fig. 1.

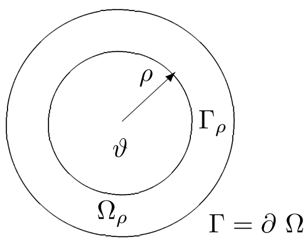

Fig. 1. The domain Ω_ρ

We denote by

$$\begin{array}{c} \Omega_\rho = \Omega \setminus \overline{B(\rho)} \subset R^2 \\ \partial\,\Omega_\rho = \Gamma \cup \Gamma_\rho \end{array} \quad (2)$$

where: Ω is a domain on the plane R^2 with a smooth boundary $\partial\,\Omega$ and

$$B_\rho = \{x : |x - \vartheta| < \rho\} \quad (3)$$

with a smooth boundary $\Gamma\rho$.

First we shall present sufficient conditions for the existence of a unique solution of the problem (1) for the case where the boundary control $v \in L^2(\Sigma)$.

For simplicity, we introduce the folowing notations:

$$Q = \Omega_\rho \times (0,T), \quad \Sigma = \Gamma \times (0,T)$$

$$E_j \triangleq ((j-1)h, jh), \; Q_j = \Omega_\rho \times E_j, \; \Sigma_j = \Gamma \times E_j$$

$$\Sigma_0 = \Gamma \times [-h, 0) \qquad \text{for} \quad j = 1, ..., K$$

For this purpose, for any pair real numbers $r, s \geq 0$ we introduce the Sobolev space $H^{r,s}(Q)$ (Lions and Magenes (1972), vol. 2, p.6) defined by

$$\begin{array}{l} H^{r,s}(Q) = \\ = H^0(0,T; H^r(\Omega_\rho)) \cap H^s(0,T; H^0(\Omega_\rho)) \end{array} \quad (4)$$

which is a Hilbert space normed by

$$\left(\int_0^T \left\| y(t) \right\|^2_{H^r(\Omega_\rho)} dt + \left\| y \right\|^2_{H^s(0,T;H^0(\Omega_\rho))} \right)^{1/2} \quad (5)$$

where: $H^s(0,T;H^0(\Omega_\rho))$ denotes the Soboles space of order s of function defined on $(0,T)$ and taking values in $H^0(\Omega_\rho)$.

The existence of a unique solution for the mixed initial-boundary value problem (1) on the cylinder Q can be proved using a constructive method, i.e. first solving (1) on the subcylinder Q_1 and in turn on Q_2 etc., until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

Using the results of (Lions and Magenes (1972), vol. 2, p. 81) we can prove the following theorem:

Theorem 1. Let y_0, Ψ_0, v and f be given with $y_0 \in H^{1/2}(\Omega_\rho)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $f \in (H^{1/2,1/4}(Q))'$. Then there exists a unique solution $y \in H^{3/2,3/4}(Q)$ for the mixed initial boundary value problem (1). Moreover, $y(\cdot, jh) \in H^{1/2}(\Omega_\rho)$ for $j = 1, ...K$.

Let us surround Γ_ρ by the circle Γ_R such that $R > \rho > 0$ (Fig.2) .

Consequently, we denote

$$\Omega_R = \Omega \setminus \overline{B(R)} \quad (6)$$

where:

$$B(R) = \{x : |x - \vartheta| < R\}. \quad (7)$$

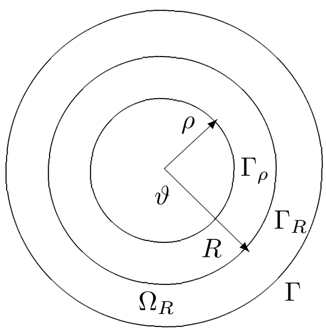

Fig. 2. The domain Ω_R

We set the non-local Neumann boundary condition on Γ_R:

$$\frac{\partial y}{\partial \eta} = A_\rho(y) \text{ on } \Gamma_R \tag{8}$$

where: A_ρ is a Steklov-Poincare operator defined in the domain $C(R,\rho) = B(R) \setminus \overline{B(\rho)}$.The operator A_ρ is a mapping of $H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$. Consequently, we consider in $\Omega_R \times (0,T)$ the following time delay parabolic equation:

$$\left.\begin{array}{lll} \frac{\partial y}{\partial t} - \Delta y = f & \text{in} & \Omega_R \times (0,T) \\ \frac{\partial y}{\partial \eta} = y(x,t-h) + v & \text{on} & \Gamma \times (0,T) \\ \frac{\partial y}{\partial \eta} = A_\rho(y) & \text{on} & \Gamma_R \times (0,T) \\ y(x,0) = y_0(x) & \text{in} & \Omega_\rho \\ y(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0) \end{array}\right\} \tag{9}$$

We shall investigate the dependence of optimal solutions on the small parameter $\rho > 0$.

The small hole $B(\rho)$ is a singular perturbation in the domain $\Omega\rho$. Consequently, the same small hole constitutes regular perturbation in the domain Ω_R.

Using the results of Sokołowski and Żochowski (2005) we obtain the following expansion for the operator A_ρ:

$$\begin{array}{l} A_\rho = A_0 + \rho^2 B + O(\rho^4) \\ \text{in the operator norm} \\ \mathcal{L}(H^{1/2}(\Gamma_R), H^{-1/2}(\Gamma_R)) \end{array} \tag{10}$$

where: the remainder $O(\rho^4)$ is uniformly bounded on bounded sets in the space $H^{1/2}(\Gamma_R)$.

Corollary 1. In the space $H^{3/2,3/4}(Q)$ the solution of the parabolic equation (for $\rho = 0$) can be represented as

$$\left.\begin{array}{lll} \frac{\partial y^0}{\partial t} - \Delta y^0 = f & \text{in} & \Omega_R \times (0,T) \\ \frac{\partial y^0}{\partial \eta} = y^0(x,t-h) + v & \text{on} & \Gamma \times (0,T) \\ \frac{\partial y^0}{\partial \eta} = A_0(y^0) & \text{on} & \Gamma_R \times (0,T) \\ y^0(x,0) = y_0(x) & \text{in} & \Omega_\rho \\ y^0(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0) \end{array}\right\} \tag{11}$$

We shall look the expansion of the solution y^ρ in $\Omega_R \times (0,T)$:

$$\begin{aligned} y^\rho &= y^0 + \rho^2 y^1 + \tilde{y} = \\ &= y^0 + \rho^2 y^1 + \rho^4 \hat{y} \end{aligned} \tag{12}$$

Consequently, the Neumann boundary condition in (9) can be rewritten as

$$\begin{aligned} \frac{\partial y^\rho}{\partial \eta} &= A_\rho(y^\rho) = \\ &= A_0(y^\rho) + \rho^2 B(y^\rho) + \rho^4 \tilde{A}(y^\rho) \end{aligned} \tag{13}$$

Substituting (12) into (13) we obtain

$$\begin{array}{l} \frac{\partial y^0}{\partial \eta} + \rho^2 B \frac{\partial y^1}{\partial \eta} + \frac{\partial \tilde{y}}{\partial \eta} = \\ = A_0(y^0 + \rho^2 y^1 + \tilde{y}) + \\ + \rho^2 B(y^0 + \rho^2 y^1 + \tilde{y}) + \rho^4 \tilde{A}(y^\rho) \end{array} \tag{14}$$

Comparing components with the same powers we get

$$\left.\begin{array}{l} \rho^0 : \frac{\partial y^0}{\partial \eta} = A_0(y^0) \\ \rho^2 : \rho^2 \frac{\partial y^1}{\partial \eta} = \rho^2 [A_0 y^1 + B y^0] \end{array}\right\} \tag{15}$$

Hence it follows the following expansion of solutions:

Let us denote by y^0 the solution of the problem (11) corresponding to a given parameter $\rho = 0$.

Subsequently, y^1 corresponding to a given parameter ρ^2 is a solution of the following equation:

$$\left.\begin{array}{lll} \frac{\partial y^1}{\partial t} - \Delta y^1 = 0 & \text{in} & \Omega_R \times (0,T) \\ \frac{\partial y^1}{\partial \eta} = y^1(x,t-h) + v & \text{on} & \Gamma \times (0,T) \\ \frac{\partial y^1}{\partial \eta} = A_0(y^1) + B(y^0) & \text{on} & \Gamma_R \times (0,T) \\ y^1(x,0) = 0 & \text{in} & \Omega_R \\ y^1(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0) \end{array}\right\} \tag{16}$$

3. PROBLEM FORMULATION. OPTIMIZATION THEOREMS.

We shall now consider the optimal boundary control problem in domains Ω_ρ and Ω_R respectively. Let us denote by $U = L^2(\Gamma \times (0,T))$ the space of controls. The time horizon T is fixed in our problem.

Let us consider in $\Omega_\rho \times (0,T)$ the following parabolic equation

$$\left.\begin{array}{lll} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in} & \Omega_\rho \times (0,T) \\ & & \text{supp } f \subset \Omega_R \times (0,T) \\ \dfrac{\partial y}{\partial \eta} = y(x,t-h) + v & \text{on} & \Gamma \times (0,T) \\ \dfrac{\partial y}{\partial \eta} = 0 & \text{on} & \Gamma_\rho \times (0,T) \\ y(x,0) = y_0(x) & \text{in} & \Omega_\rho \\ & \text{supp} & y_0 \subset \Omega_R \\ y(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0) \end{array}\right\} \quad (17)$$

The performance functional is given by

$$I(v) = \frac{1}{2}\int_{\Omega_R} |y(x,T;v) - z_d|^2 dx + \frac{\alpha}{2}\int_0^T \int_\Gamma |v|^2 dx dt \quad (18)$$

Finally, we assume the following constraints on the control $v \in U_{ad}$:

$$U_{ad} = \{v \in L^2(\Gamma \times (0,T)), 0 \le v(x,t) \le 1\} \quad (19)$$

Subsequently, we consider in $\Omega_R \times (0,T)$ the following parabolic time delay equation

$$\left.\begin{array}{lll} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in} & \Omega_R \times (0,T) \\ \dfrac{\partial y}{\partial \eta} = y(x,t-h) + v & \text{on} & \Gamma \times (0,T) \\ \dfrac{\partial y}{\partial \eta} = A_\rho(y) & \text{on} & \Gamma_R \times (0,T) \\ y(x,0) = y_0(x) & \text{in} & \Omega_R \\ y(x,t') = \Psi_0(x,t') & \text{in} & \Gamma \times [-h,0) \end{array}\right\} \quad (20)$$

The performance functional and constraints on the control are given by (18) and (19).

Result: The solution of the problem (20) (in the domain Ω_R) is a restriction of the solution of the problem (17) (in the domain Ω_ρ) to Ω_R. Hence, we have the possibility of replacing the singular perturbation of the domain $B(\rho)$ by the regular perturbation on the boundary Γ_R in a smaller domain Ω_R. Consequently, we shall analyse the optimal boundary control problem (18)-(20) in the domain Ω_R. Moreover, we assume the fixed parameter $\rho > 0$. The solving of the formulated optimal control problem is equivalent to seeking a $v_0 \in U_{ad}$ such that $I(v_0) \le I(v)\ \forall v \in U_{ad}$.

From Lions' scheme (Theorem 1.3 Lions (1971), p. 10) it follows that for $\alpha > 0$ a unique optimal control v_0 is characterized by the following condition

$$I'(v_0)(v - v_0) \ge 0 \quad \forall v \in U_{ad} \quad (21)$$

Using the form of the performance functional (18) we can express (21) in the following form:

$$\int_{\Omega_R} (y(x,T;v_0) - z_d)(y(x,T;v) - y(x,T;v_0)dx + \frac{\alpha}{2}\int_0^T \int_\Gamma v_0(v - v_0) dx dt \ge 0 \quad \forall v \in U_{ad} \quad (22)$$

To simplify (22), we introduce the adjoint equation and for every $v \in U_{ad}$. we define the adjoint variable $p = p(v) = p(x,t;v)$ as the solution of the following equation

$$\left.\begin{array}{ll} -\dfrac{\partial p}{\partial t} - \Delta p = 0 & \text{in } \Omega_R \times (0,T) \\ \dfrac{\partial p}{\partial \eta} = p(x,t+h) & \text{on } \Gamma \times (0,T-h) \\ \dfrac{\partial p}{\partial \eta} = 0 & \text{on } \Gamma \times (T-h,T) \\ \dfrac{\partial p}{\partial \eta} = A_\rho(p) & \text{on } \Gamma_R \times (0,T) \\ p(x,T;v) = y(x,T;v) \ -z_d \ \text{ in } \Omega_R & \end{array}\right\} \quad (23)$$

Theorem 2. Let the hypothesis of Theorem 1 be satisfied. Then for given $z_d \in L^2(\Omega_R)$ and any $v_0 \in U_{ad}$, there exists a unique solution $p(v_0) \in H^{3/2,3/4}(Q)$ for the problem (23).

We simplify (22) using the adjoint equation (23). Consequently, after transormations we obtain the following formula

$$\int_0^T \int_\Gamma (p + \alpha v_0)(v - v_0) dx dt \ge 0 \quad \forall v \in U_{ad} \quad (24)$$

Theorem 3. For the problem (20) with the performance functional (18) with $z_d \in L^2(\Omega_R)$ and $\alpha > 0$, and with constraints on the control (19), there existsts a unique optimal control v_0 which satisfies the maximum condition (24). Moreover, $v_0 = P_{U_{ad}}(-\frac{1}{\alpha}p)$ where $P_{U_{ad}}$ is a projective operator.

4. THE SENSIVITY OF OPTIMAL CONTROLS

Theorem 4. We have the following expansion of the optimal control in $L^2(\Gamma \times (0,T))$, with respect to the small parameter,

$$v_\rho = v_0 + \rho^2 q + o(\rho^2) \tag{25}$$

for $\rho > 0$.

Moreover, we assume that ρ is a sufficiently small. The function q in (25) is a optimal solution of the following optimal control problem:

The state equation

$$\left.\begin{array}{ll} \dfrac{\partial w}{\partial t} - \Delta w = 0 & \text{in } \Omega_R \times (0,T) \\ \dfrac{\partial w}{\partial \eta} = w(x,t-h) + q & \text{on } \Gamma \times (0,T) \\ \dfrac{\partial w}{\partial \eta} = A_0(w) + B(y^0) & \text{on } \Gamma_R \times (0,T) \\ w(x,0) = 0 & \text{in } \Omega_R \\ w(x,t') = \Psi_0(x,t') & \text{on } \Gamma \times [-h,0) \end{array}\right\} \tag{26}$$

where: $w = y^1$.

The performance functional

$$I(u) = \frac{1}{2}\int_{\Omega_R} |w(T,x)|^2 dx + \frac{\alpha}{2}\int_0^T\int_\Gamma |u|^2 dx dt \tag{27}$$

The adjoint equation

$$\left.\begin{array}{ll} -\dfrac{\partial z}{\partial t} - \Delta z = 0 & \text{in } \Omega_R \times (0,T) \\ \dfrac{\partial z}{\partial \eta} = z(x,t+h) & \text{on } \Gamma \times (0,T-h) \\ \dfrac{\partial z}{\partial \eta} = 0 & \text{on } \Gamma \times (T-h,T) \\ \dfrac{\partial z}{\partial \eta} = A_0(z) + B(p^0) & \text{on } \Gamma_R \times (0,T) \\ z(x,T) = w(x,T) & \text{in } \Omega_R \end{array}\right\} \tag{28}$$

where: $z = p^1$.

Then, the optimal control q is characterized by

$$\begin{aligned} &\int_{\Omega_R} w(x,T;q)(w(x,T;u) - w(x,T;q))dx + \\ &+\int_0^T\int_\Gamma q(u-q)dxdt \geq 0 \qquad \forall u \in S_{ad} \end{aligned} \tag{29}$$

where: S_{ad} is a set of admissible controls such that

$$\begin{aligned} S_{ad} = \Big\{ u \in L^2(\Gamma \times (0,T)) \Big| \\ & u(x,t) \geq 0 \text{ on the set} \\ & \qquad E_0 = \{(x,t)|v_0(x,t) = 0\}, \\ & u(x,t) < 0 \text{ on the set} \\ & \qquad E_1 = \{(x,t)|v_0(x,t) = 1\}, \\ & \int_0^T\int_\Gamma (p_0 + \alpha v_0) u dx dt = 0 \Big\}, \end{aligned} \tag{30}$$

where:

p_0 is a adjoint state for $\rho = 0$,

v_0 is a optimal solution for $\rho = 0$ such that

$0 \leq v_0(x,t) \leq 1$.

We simplify (29) using the adjoint equation (28). After transformations we obtain the following maximum condition

$$\int_0^T\int_\Gamma (z + \alpha q)(u - q) dx dt \geq 0 \quad \forall u \in S_{ad} \tag{31}$$

Theorem 5. For the time delay parabolic problem

$$\left.\begin{array}{ll} \dfrac{\partial w}{\partial t} - \Delta w = 0 & \text{in } \Omega_R \times (0,T) \\ \dfrac{\partial w}{\partial \eta} = w(x,t-h) + u & \text{on } \Gamma \times (0,T) \\ \dfrac{\partial w}{\partial \eta} = A_0(w) + B(y^0) & \text{on } \Gamma_R \times (0,T) \\ w(x,0) = 0 & \text{in } \Omega_R \\ w(x,t') = \Psi_0(x,t') & \text{in } \Gamma \times [-h,0) \end{array}\right\} \tag{32}$$

with the performance functional (27) with $w(T) \in L^2(\Omega_R)$ and $\alpha > 0$, and with constraints on the control (30), there exists a unique optimal control q which satisfies the maximum condition (31).

5. CONCLUSIONS

The results presented in the paper can be treated as a generalization of the results obtained in Ref. Sokołowski and Żochowski (2005) onto the case of parabolic systems with boundary condition involving time delays.

In this paper we have considered the mixed initial boundary value problems of parabolic type.

We can also consider similar optimal control problems for time delay hyperbolic systems.

The ideas mentioned above will be developed in forthcoming papers.

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References

L. Jackowska, J. Sokołowski, and A. Żochowski. Topological optimization and inverse problems. *Computer Assisted Mechanics and Engineering Sciences*, 10:163–176, 2003.

L. Jackowska, J. Sokołowski, A. Żochowski, and A. Henrot. On numerical solution of shape inverse problems. *Computational Optimization and Applications*, 23:231–255, 2002.

I. Lasiecka and J. Sokołowski. Sensitivity analysis of constrained optimal control problem for wave equation. *SIAM Journal on Control and Optimization*, 29:1128–1149, 1991.

J.L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations*. Springer-Verlag, Berlin-Heidelberg, 1971.

J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications –Vols. 1 adn. 2*. Springer-Verlag, Berlin-Heidelberg, 1972.

K. Malanowski. *Stability and sensitivity analysis for optimal control problems with control-state constraints*. Disertations Math. (Rozprawy Mat.), Warsaw, 2001.

K. Malanowski and J. Sokołowski. Sensitivity of solutions to convex, control constrained optimal control problems for distributed parameter systems. *Journal of Mathematical Analysis and Applications*, 120:240–263, 1986.

V. Maz'ya, S. Nazarov, and B. Plamenevskij. *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains -vol.1*. Birkhäuser Verlag, Basel, 2000.

S. A. Nazarov. Asymptotic conditions at a point, self adjoint extensions of operators and the method of matched asymptotic expansions. *American Mathematical Society Translations (2)*, 198:77–125, 1999.

S. A. Nazarov, S. A. Slutskij, and J. Sokołowski. Topological derivative of the energy functional due to formation of a thin ligament on a spatial body. *Les prépublications de l'Institut Élie Cartan*, 14, 2004.

S. A. Nazarov and J. Sokołowski. Asymptotic analysis of shape functionals. *Journal de Mathématiques pures et appliquées*, 82:125–196, 2003a.

S. A. Nazarov and J. Sokołowski. Self adjoint extensions for the neumann laplacian in application to shape optimization. *Les prépublications de l'Institut Élie Cartan*, 9, 2003b.

S. A. Nazarov and J. Sokołowski. Self adjoint extensions of differential operators in application to shape optimization. *Comptes Rendus Mecanique*, 331:667–672, 2003c.

S. A. Nazarov and J. Sokołowski. The topological derivative of the dirichlet integral due to formation of a thin ligament. *Siberian Math. J.*, 45: 341–355, 2004.

M. Rao and J. Sokołowski. Tangent sets in Banach spaces and applications to variational inequalities. *Les prépublications de l'Institut Élie Cartan*, 42, 2000.

J. Sokołowski. Differential stability of solutions to constrained optimization problems. *Appl. Math. Optim.*, 13:97–115, 1985.

J. Sokołowski. Sensitivity analysis of control constrained optimal control problems for distributed parameter systems. *SIAM J. Control and Optimization*, 25:1542–1556, 1987.

J. Sokołowski. Shape sensitivity analysis of boundary optimal control problems for parabolic systems. *SIAM J. Control and Optimization*, 26:763–787, 1988.

J. Sokołowski and A. Żochowski. On topological derivative in shape optimization. *SIAM J. Control and Optimization*, 37:1251–1272, 1999a.

J. Sokołowski and A. Żochowski. Topological derivative for optimal control problems. *Control and Cybernetics*, 28:611–626, 1999b.

J. Sokołowski and A. Żochowski. Topological derivatives for elliptic problems. *Inverse Problems*, 1:123–134, 1999c.

J. Sokołowski and A. Żochowski. Topological derivatives of shape functionals for elasticity systems. *Mechanics of Structures and Machines*, 29:333–351, 2001.

J. Sokołowski and A. Żochowski. Optimality conditions for simultaneous topology and shape optimization. *SIAM J. Control and Optimization*, 42:1198–1221, 2003.

J. Sokołowski and A. Żochowski. Topological derivatives for obstacle problems. *Les prépublications de l'Institut Élie Cartan*, 12, 2005.

J. Sokołowski and J. P. Zolesio. *Introduction to Shape Optimization. Shape Sensitivity Analysis*. Springer Verlag, Berlin-Heidelberg, 1992.