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POLE-BY-POLE SHIFTING VIA A LINEAR-QUADRATIC REGULATION

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Abstract: The linear-quadratic regulator and pole placement techniques are considered for designing continuous-time multivariable control systems. The proposed method combines the two approaches in a particular manner. The weighting matrices for the linear-quadratic optimization are constructed corresponding to a set of prescribed eigenvalues. In fact, a single eigenvalue (or a pair of complex conjugate eigenvalues) can be shifted at a time, leaving the remaining eigenvalues at their original positions. The simultaneous knowledge of the weights and the associated closed-loop eigenvalues provides the designer with the opportunity of interaction in both directions. Thereby eigenvalues located in undesired positions can be shifted to more suitable ones. The area into which each eigenvalue can be shifted is described in detail. The allowable shifts may result in a faster and dampening feedback.

Keywords: Control system design; continuous-time systems; pole placement; linear-quadratic regulator; successive eigenvalue relocation.

1 INTRODUCTION

Linear-quadratic regulation and pole placement (or eigenvalue assignment) are two popular methods for the design of linear control systems. The former constructs a state feedback gain so as to stabilize the system and minimize a quadratic cost, which defines a relative importance of various state variables and control inputs through given weighting matrices. The latter then selects a state feedback gain so as to achieve a prescribed set of eigenvalues.

The optimal linear-quadratic design has several nice features. In particular, the closed-loop system enjoys certain robustness properties provided the weighting matrices satisfy certain positivity conditions (Anderson and Moore 1990). The transient behavior of the closed-loop system, however, is difficult to determine in advance since there is no simple relation between the weighting matrices and the closed-loop eigenvalues. To get a good transient response, the weights are often determined iteratively through trial and error.

Alternatively, pole placement methods have the advantage that the closed-loop eigenvalues can be specified directly. Therefore the transient phenomena can de addressed in a direct manner. A drawback is that many different feedback gains can lead to the same pole pattern when the system has several inputs and these gains can produce very different transients (Antsaklis and Michel 1997).

Attempts to combine the two methods are of an early date. Results exist on optimal control with eigenvalues restricted to a specified region of the complex plane, namely a semi-plane (Anderson and Moore (1969), a disk (Furuta and Kim 1987), a sector (Hench *et al.* 1998), or a hyperbolic region (Kawasaki and Shimemura 1983). Optimal control with exactly prescribed eigenvalues is more difficult. Various results reflect various approaches to seeking a relationship between the weighting matrices and eigenvalue locations (Solheim 1972), (Amin 1985), (Alexandridis and Galanos 1987), (Seif 1989), (Sugimoto and Yamamoto 1989), (Duplaix *et al.* 1994), (Franceschi *et al.* 1995), (Kučera and Kraus 1999), (Kraus and Kučera 1999), and (Cigler 2009).

This paper is inspired by (Kraus and Kučera 1999) and is a presentation of the Master Thesis (Cigler 2009) in which some of the restrictions of the earlier results are relaxed and the regions of target eigenvalue locations are explicitly described. The method combines the linear-quadratic optimization with pole placement in a particular manner. The weighting matrices of the optimal problem are constructed so as to shift a single eigenvalue (or a pair of complex conjugate eigenvalues) to a prescribed position while leaving the remaining eigenvalues at their original positions. The process can be repeated until a desired pole pattern is achieved. The simultaneous knowledge of the weights and the associated closed-loop eigenvalues provides the designer with the opportunity of interaction in both directions.

2 PRELIMINARIES

Let us review the linear-quadratic regulator problem and fix the notation to be used throughout the paper.

Given a linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0 \tag{1}$$

we seek a control law

$$u(t) = Fx(t)$$

that stabilizes the feedback system and minimizes a quadratic cost of the form

$$\int_0^\infty (x^T Qx + u^T Ru)dt \tag{2}$$

for every initial state $x(0)$. The matrices Q and R are symmetric with $Q = C^T C \geq 0$ and $R > 0$.

We suppose that the pair (A, B) is stabilizable and the pair (A, C) is detectable. Then there exists a unique symmetric matrix solution P of the algebraic Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{3}$$

such that $P \geq 0$ and the state feedback gain matrix

$$F = -R^{-1}B^T P \tag{4}$$

stabilizes the feedback system

$$\dot{x}(t) = (A + BF)x(t) \tag{5}$$

while minimizing the cost (3).

Consider the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}. \tag{6}$$

The eigenvalues of H are symmetrically distributed with respect to the imaginary axis. Let T be a similarity transformation that brings H to its Jordan form arranged so that

$$H = T\begin{bmatrix} J & 0 \\ 0 & -J^T \end{bmatrix}T^{-1}. \tag{7}$$

Under the detectability assumption, H has no pure imaginary eigenvalue and J can be taken to be a stable matrix. Decompose T compatibly,

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

Under the stabilizability assumption, T_{11} is nonsingular and

$$P = T_{21}T_{11}^{-1}.$$

It follows from (4), (6) and (7) that

$$A + BF = A - BR^{-1}B^T = T_{11}JT_{11}^{-1}$$

so that the closed-loop system matrix (5) is similar to the matrix J.

3 SINGLE EIGENVALUE RELOCATION

The linear-quadratic regulator imposes the eigenvalues of the closed-loop system and, accordingly, it can be considered a special case of the eigenvalue assignment (or pole placement) design problem. Given A and B, the choice of Q and R achieves a certain pattern of the eigenvalues of H, which in turn define the closed-loop system eigenvalues.

In order to relate the two design techniques more closely, we shall investigate the possibility of relocating a single eigenvalue at a time, leaving the remaining eigenvalues at their original positions. This can indeed be achieved by an appropriate choice of the weighting matrices Q and R. For the sake of exposition, we shall consider the two cases as follows.

3.1 *The case of a real simple eigenvalue*

Let T be a similarity transformation that brings A to its Jordan form,

$$\widetilde{A} = T^{-1}AT, \quad \widetilde{B} = T^{-1}B \tag{8}$$

and suppose that $\widetilde{A}$ is diagonal. The columns of T, denoted $v_1, v_2, ..., v_n$, are the right eigenvectors of A while the rows of T^{-1}, denoted as $w_1^T, w_2^T, ..., w_n^T$, are the left eigenvectors of A.

Choose one controllable eigenvalue, say λ_1, of A to be shifted and exhibit it in the Jordan form as follows

$$\widetilde{A}=\begin{bmatrix}\lambda_1 & 0\\ 0 & J_1\end{bmatrix},\quad \widetilde{B}=\begin{bmatrix}b_1^T\\ \times\end{bmatrix}, \tag{9}$$

where b_1^T is the first row of matrix $\widetilde{B}$ and $\times$ indicates the remaining entries.

Take the weighting matrix Q as

$$Q=(T^{-1})^T\widetilde{Q}T^{-1}, \tag{10}$$

where

$$\widetilde{Q}=\begin{bmatrix}q_1 & 0\\ 0 & 0\end{bmatrix} \tag{11}$$

with $q_1\ge 0$ a real parameter, and select the weighting matrix R so that $b_1^T R^{-1} b_1 = 1$. Make an inspired guess that the optimal solution matrix P of the Riccati equation (3) is

$$P=(T^{-1})^T\widetilde{P}T^{-1}, \tag{12}$$

where

$$\widetilde{P}=\begin{bmatrix}p_1 & 0\\ 0 & 0\end{bmatrix} \tag{13}$$

for some real constant $p_1\ge 0$. Substituting (8), (10) and (12) into (3) gives

$$T(\widetilde{P}\widetilde{A}+\widetilde{A}^T\widetilde{P}-\widetilde{P}\widetilde{B}\widetilde{R}^{-1}\widetilde{B}^T\widetilde{P}+\widetilde{Q})T^{-1}=0\,.$$

Using (9), (11) and (13), one reduces the Riccati equation to a scalar equation for p_1, namely

$$p_1^2-2\lambda_1 p_1-q_1=0\,,$$

which can readily be solved.

Let μ_1 be the desired position to which the eigenvalue λ_1 is to be shifted and suppose that μ_1 is stable. We shall first analyze which positions for μ_1 are eligible and which matrices Q realize the shift. Consider the Hamiltonian matrix (6),

$$H=\begin{bmatrix}T & 0\\ 0 & (T^{-1})^T\end{bmatrix}\begin{bmatrix}\widetilde{A} & -\widetilde{B}R^{-1}\widetilde{B}^T\\ -\widetilde{Q} & -\widetilde{A}^T\end{bmatrix}\begin{bmatrix}T^{-1} & 0\\ 0 & T^T\end{bmatrix},$$

and calculate

$$\det(sI-H)=\det\begin{bmatrix}sI-\widetilde{A} & \widetilde{B}R^{-1}\widetilde{B}^T\\ \widetilde{Q} & sI+\widetilde{A}^T\end{bmatrix}$$

$$=\det\left[\begin{array}{cc:cc} s-\lambda_1 & 0 & 1 & \times\\ 0 & sI-J_1 & \times & \times\\ \hdashline q_1 & 0 & s+\lambda_1 & 0\\ 0 & 0 & 0 & s+J_1^T\end{array}\right]$$

$$=\det(sI-J_1)\,\det(sI+J_1^T)\,\det(sI-H_1)\,,$$

where

$$H_1=\begin{bmatrix}\lambda_1 & -1\\ -q_1 & -\lambda_1\end{bmatrix}$$

and where $\times$ indicates the remaining entries. It follows that all the eigenvalues of A but λ_1 remain unchanged and the shift of λ_1 to μ_1 requires that

$$\det(sI-H_1)=(s-\mu)(s+\mu)\,,$$

that is,

$$s^2-(\lambda_1^2+q_1)=s^2-\mu_1^2\,.$$

We conclude that $\mu_1\le -|\lambda_1|$ since $q_1\ge 0$. The eigenvalue can only be shifted to the left. Note that when λ_1 is not stable it is shifted to the left of its stable image $-\lambda_1$.

To summarize, one can pick any real eigenvalue λ_1 of A and shift it to a desired position $\mu_1\le -|\lambda_1|$ while keeping the remaining eigenvalues unchanged. This can be done by solving a simple linear-quadratic regulator problem. The problem has an explicit solution in terms of the left eigenvector w_1^T of A that is associated with λ_1. Taking

$$Q=w_1(\mu_1^2-\lambda_1^2)\,w_1^T$$

and selecting R such that $w_1^T BR^{-1}B^T w_1=1$ in (2), one obtains

$$P=w_1(\lambda_1-\mu_1)\,w_1^T\,.$$

The feedback gain matrix F that accomplishes this task is given by (4).

The process can be repeated for each eigenvalue *ad libitum*. We note, however, that eigenvalues can only be shifted to the left due to the special structure of Q.

3.2 *The case of a real multiple eigenvalue*

Now suppose that the controllable eigenvalue of A to be shifted, call it again λ_1, is real but generates a Jordan block of size k,

$$\widetilde{A}=\left[\begin{array}{cccc:c}\lambda_1 & & & & \\ 1 & \lambda_1 & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda_1 & \\ \hdashline & & & & J_k\end{array}\right].$$

We claim that the result obtained in Subsection 4.1 holds in this case also. Indeed, the choice of Q as

$$Q=w_1(\mu_1^2-\lambda_1^2)\,w_1^T$$

and that of R such that $w_1^T BR^{-1}B^T w_1=1$ leads to the solution matrix P of the Riccati equation (3) in the form

$$P = w_1(\lambda_1 - \mu_1)\, w_1^T ,$$

thus resulting in a shift of λ_1 to a position $\mu_1 \le -|\lambda_1|$. The remaining eigenvalues of A keep their original positions. In particular, λ_1 remains an eigenvalue of A but it generates a Jordan block of size $k-1$.

Therefore, the effect of the feedback gain matrix

$$F = -B^T w_1(\lambda_1 - \mu_1)\, w_1^T$$

on system (1) is to split the Jordan block of λ_1 into a single eigenvalue μ_1 and a smaller block of λ_1. This process can be continued, resulting in a spectrum of k eigenvalues $\mu_1, \mu_2, ..., \mu_k$ positioned to the left of the value $-|\lambda_1|$.

4 RELOCATION OF A COMPLEX CONJUGATE PAIR OF EIGENVALUES

Suppose that A has a pair of simple, complex conjugate eigenvalues, say $\lambda_1 = \lambda$ and $\lambda_2 = \overline{\lambda}$, which are controllable and are to be shifted simultaneously to obtain a new complex conjugate pair of stable eigenvalues $\mu_1 = \mu$ and $\mu_2 = \overline{\mu}$. In this case we have

$$\widetilde{A} = \begin{bmatrix} \Lambda_2 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_2^T \\ \times \end{bmatrix}, \qquad (14)$$

where

$$\Lambda_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \qquad (15)$$

and where B_2^T denotes the first two rows of $\widetilde{B}$ and $\times$ indicates the remaining entries.

The eigenvectors of A that are associated with the eigenvalues $\lambda, \overline{\lambda}$ (the first two columns of T and the first two rows of T^{-1}) are denoted as $v_1 = v, v_2 = \overline{v}$ and $w_1^T = \overline{w}^T, w_2^T = w^T$.

Take the weighting matrix Q as

$$Q = (\overline{T}^{-1})^T \widetilde{Q} T^{-1}, \qquad (16)$$

where

$$\widetilde{Q} = \begin{bmatrix} Q_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad (17)$$

and $Q_2 \ge 0$ is a Hermitian 2×2 matrix parameter. As the first two columns of T are complex conjugate of each other, Q_2 will have equal diagonal entries,

$$Q_2 = \begin{bmatrix} q & q_{12} \\ \overline{q}_{12} & q \end{bmatrix} \qquad (18)$$

for a real q and a complex q_{12} that satisfy $q \ge |q_{12}|$.

Select the weighting matrix R so that

$$B_2^T R^{-1} \overline{B}_2 = \begin{bmatrix} 1 & \overline{\omega} \\ \omega & 1 \end{bmatrix} := \Omega_2 \qquad (19)$$

for a complex ω such that $|\omega| \le 1$.

Make an inspired guess that the optimal solution matrix P of the Riccati equation (3) is

$$P = (\overline{T}^{-1})^T \widetilde{P} T^{-1}, \qquad (20)$$

where

$$\widetilde{P} = \begin{bmatrix} P_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad (21)$$

for some 2×2 Hermitian matrix $P_2 \ge 0$ having equal diagonal entries. Substituting (8), (16) and (20) into (3) yields

$$T(\widetilde{P}\widetilde{A} + \overline{\widetilde{A}}^T \widetilde{P} - \widetilde{P}\widetilde{B}\widetilde{R}^{-1}\overline{\widetilde{B}}^T \widetilde{P} + \widetilde{Q})T^{-1} = 0 .$$

Using (14), (17), (19) and (21), one reduces the Riccati equation to

$$P_2\Lambda_2 + \overline{\Lambda}_2^T P_2 - P_2\Omega_2 P_2 + Q_2 = 0, \qquad (22)$$

to be solved for P_2.

For a single eigenvalue, only a left shift is possible. The situation is more involved in the case of shifting a pair of eigenvalues. The relevant quantities are related by the 4×4 Hamiltonian matrix

$$H_2 = \begin{bmatrix} \Lambda_2 & -\Omega_2 \\ -Q_2 & -\overline{\Lambda}_2^T \end{bmatrix}$$

whose eigenvalues are to equal $\mu, \overline{\mu}$ and $-\mu, -\overline{\mu}$. Substituting from (15), (18) and (19), we obtain

$$\det(sI - H_2) = \det \left[\begin{array}{cc:cc} s-\lambda & 0 & 1 & \overline{\omega} \\ 0 & s-\overline{\lambda} & \omega & 1 \\ \hdashline q & q_{12} & s+\overline{\lambda} & 0 \\ \overline{q}_{12} & q & 0 & s+\lambda \end{array}\right]$$

$$= s^4 - 2(\mathrm{Re}\,\lambda^2 + q + \mathrm{Re}\,\omega q_{12})s^2 + |\lambda|^4 + 2|\lambda|^2 q + 2\,\mathrm{Re}\,\lambda^2\omega q_{12} + (1-|\omega|^2)(q^2 - |q_{12}|^2).$$

The intended shift calls for

$$\det(sI - H_2) = s^4 - 2\mathrm{Re}\,\mu^2 s^2 + |\mu|^4$$

and the region into which $\lambda, \overline{\lambda}$ are allowed to be shifted is determined by the equalities

$$\mathrm{Re}\,\mu^2 = \mathrm{Re}\,\lambda^2 + q + \mathrm{Re}\,\omega q_{12} \qquad (23)$$

$$|\mu|^4 = |\lambda|^4 + 2|\lambda|^2 q + 2\,\mathrm{Re}\,\lambda^2\omega q_{12} + (1-|\omega|^2)(q^2 - |q_{12}|^2) . \qquad (24)$$

The shape of the target region for $\mu, \bar{\mu}$ depends on λ and ω. To visualize the region, we denote $x = \operatorname{Re}\mu$, $y = \operatorname{Im}\mu$ so as to have

$$\operatorname{Re}\mu^2 = x^2 - y^2, \quad |\mu|^4 = (x^2 + y^2)^2$$

and proceed by fixing the values of ω as follows.

4.1 *The case of* $|\omega| = 1$

In this case Ω is a rank-one singular matrix, which happens for single-input systems. Equations (23) and (24) read

$$x^2 - y^2 = \operatorname{Re}\lambda^2 + q + \operatorname{Re}\omega q_{12} \qquad (25)$$

$$(x^2 + y^2)^2 = |\lambda|^4 + 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2 \omega q_{12}\,. \qquad (26)$$

We observe that these equations are linear in q and are to be solved for some real $q \geq |q_{12}|$.

Therefore suppose that $q \geq |q_{12}|$. Then

$$|\operatorname{Re}\omega q_{12}| \leq |\omega q_{12}| \leq |q_{12}| \leq q$$

and

$$|\operatorname{Re}\lambda^2 \omega q_{12}| \leq |\lambda^2 \omega q_{12}| \leq |\lambda^2||q_{12}| \leq |\lambda^2| q\,.$$

In view of that,

$$q + \operatorname{Re}\omega q_{12} \geq 0, \quad 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2 \omega q_{12} \geq 0$$

and (25), (26) yield the inequalities

$$x^2 - y^2 \geq \operatorname{Re}\lambda^2 \qquad (27)$$

$$x^2 + y^2 \geq |\lambda|^2. \qquad (28)$$

Observe that (27) represents either the left half-plane interior of the equilateral hyperbola

$$x^2 - y^2 \geq \operatorname{Re}\lambda^2, \qquad (29)$$

or the left half-plane exterior of the conjugated hyperbola

$$y^2 - x^2 \leq -\operatorname{Re}\lambda^2, \qquad (30)$$

or the sector delineated by their asymptotes

$$y \geq x, \quad y \leq -x, \qquad (31)$$

depending on the sign of $\operatorname{Re}\lambda^2$. The real and imaginary axes of the above hyperbolas equal the square root of $|\operatorname{Re}\lambda^2|$.

Inequality (28) represents the exterior of a circle with radius $|\lambda|$, centered at the origin.

Figures 1 – 4 visualize as shaded areas the attainable regions for the eigenvalues $\lambda = 2j$, $\lambda = -1 + 2j$, $\lambda = -2 + 2j$, $\lambda = 3 + 2j$ and for $|\omega| = 1$. Note that the equality $q = |q_{12}|$ holds along the hyperbolas as well as the circle.

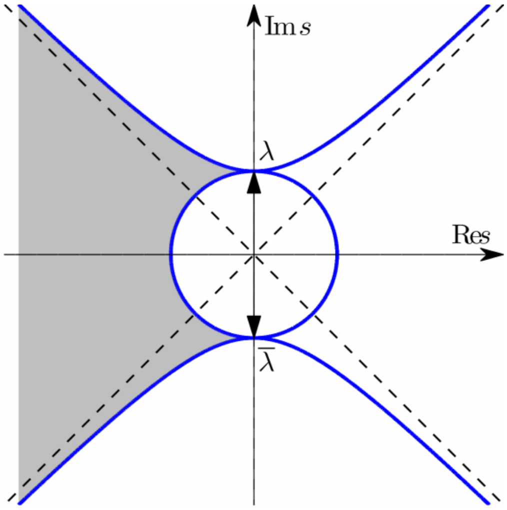

Figure 1. The allowable target region for $\lambda = 2j$ and for $|\omega| = 1$.

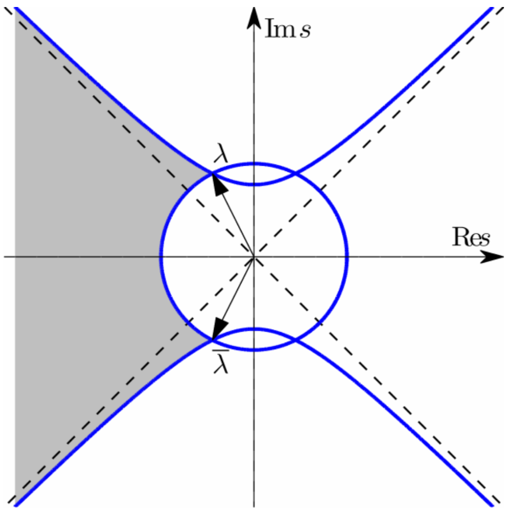

Figure 2. The allowable target region for $\lambda = -1 + 2j$ and for $|\omega| = 1$.

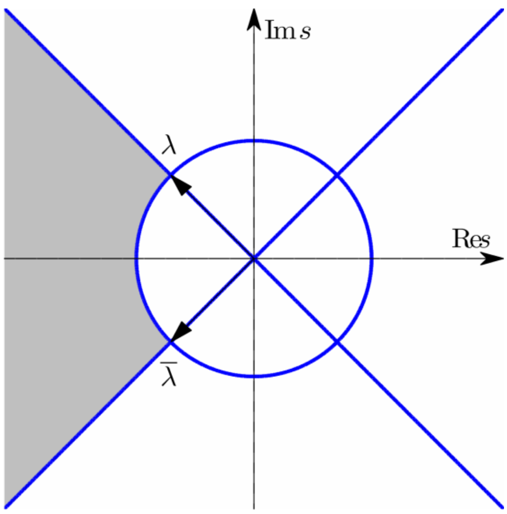

Figure 3. The allowable target region for $\lambda = -2 + 2j$ and for $|\omega| = 1$.

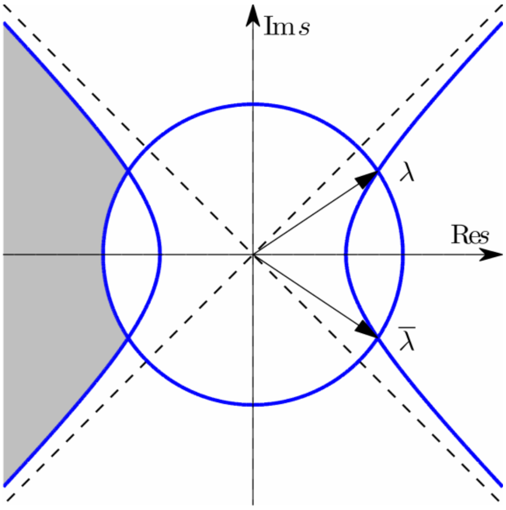

Figure 4. The allowable target region for $\lambda = 3 + 2j$ and for $|\omega| = 1$.

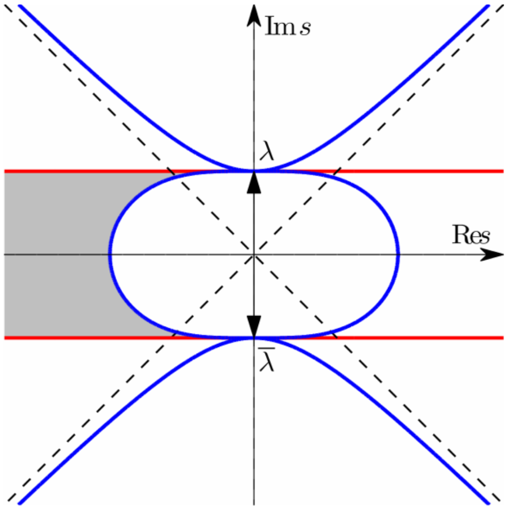

Figure 5. The allowable target region for $\lambda = 2j$ and for $\omega = 0$.

4.2 *The case of* $\omega = 0$

In this case Ω is the identity matrix. Equations (23) and (24) read

$$x^2 - y^2 = \mathrm{Re}\,\lambda^2 + q \tag{32}$$

$$(x^2 + y^2)^2 = |\lambda|^4 + 2|\lambda|^2 q + q^2 - |q_{12}|^2. \tag{33}$$

We observe that these equations are quadratic in q and are to be solved for some real q and a complex q_{12} such that $q \geq |q_{12}|$. It follows from (32) that q is real as long as (33) is satisfied for some q_{12}. Write (33) in the form

$$|q_{12}|^2 = (q + |\lambda|^2)^2 - (x^2 + y^2)^2.$$

In view of $|q_{12}| \geq 0$ this equation implies the inequality

$$x^2 + y^2 \leq q + |\lambda|^2.$$

Substituting for q from (32), one obtains

$$2y^2 \leq |\lambda|^2 - \mathrm{Re}\,\lambda^2$$

or equivalently

$$y^2 \leq \mathrm{Im}^2 \lambda. \tag{34}$$

On the other hand, the condition $q \geq |q_{12}|$ turns (33) into the inequality

$$(x^2 + y^2)^2 \geq |\lambda|^4 + 2|\lambda|^2 + q^2.$$

Substituting for q from (32), one obtains

$$(x^2 + y^2)^2 - 2|\lambda|^2 (x^2 - y^2) + |\lambda|^4 \geq 4|\lambda|^2 \,\mathrm{Im}^2 \lambda. \tag{35}$$

Observe that equation (34) defines a strip of width $2|\mathrm{Im}\lambda|$ along the real axis while equation (35) represents the exterior of a Cassini oval with foci at the

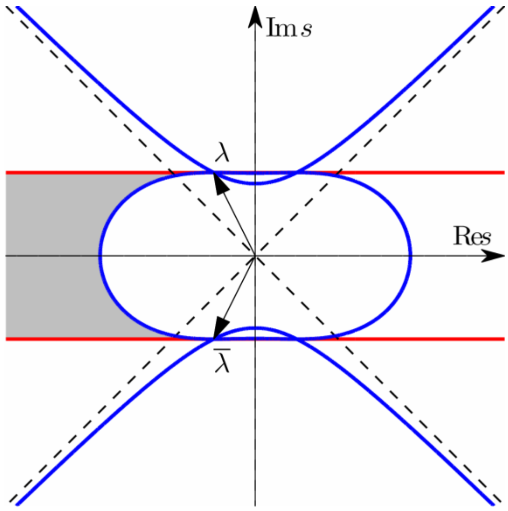

Figure 6. The allowable target region for $\lambda = -1 + 2j$ and for $\omega = 0$.

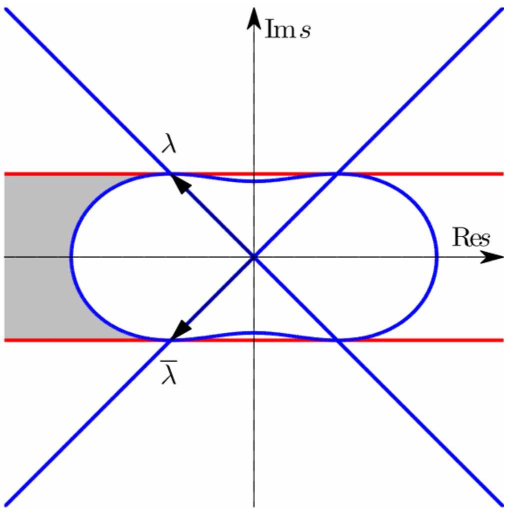

Figure 7. The allowable target region for $\lambda = -2 + 2j$ and for $\omega = 0$.

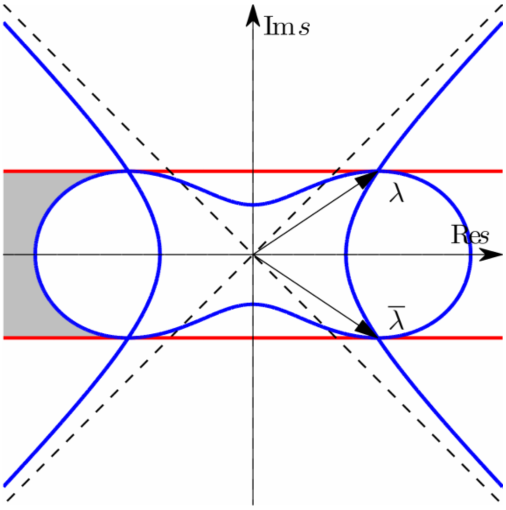

Figure 8. The allowable target region for $\lambda = 3 + 2j$ and for $\omega = 0$.

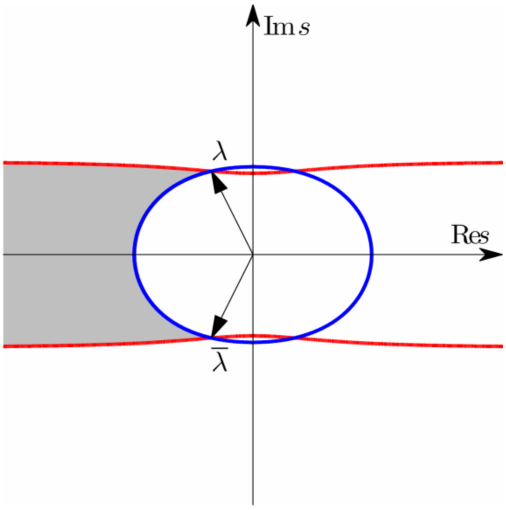

Figure 9. The allowable target region for $\lambda = -1 + 2j$ and for $|\omega| = 0.5$.

points $(x, y) = (|\lambda|, 0)$ and $(x, y) = (-|\lambda|, 0)$. The shape of the Cassini oval depends on the value of $4(\mathrm{Im}^2\lambda)/|\lambda|^2$. Thus the real part of the eigenvalues λ, $\bar{\lambda}$ can only be shifted to the left outside the oval while their imaginary parts cannot be increased.

Figures 5 – 8 visualize the allowable target regions – the shaded areas – for the eigenvalues $\lambda = 2j$, $\lambda = -1 + 2j$, $\lambda = -2 + 2j$, $\lambda = 3 + 2j$ and for $\omega = 0$. The ovals are shown in blue whereas the strip boundaries are shown in red. Note that $q = |q_{12}|$ along the blue boundary curves while $q_{12} = 0$ along the red boundary curves.

4.3 *The case of* $0 < |\omega| < 1$

In this case Ω is a general rank-two matrix. The shape of the target region can be investigated from (23) and (24) while considering the conditions for a real q and a complex q_{12} to exist such that $q \geq |q_{12}|$.

Equations (23) and (24) read

$$x^2 - y^2 = \mathrm{Re}\,\lambda^2 + q + \mathrm{Re}\,\omega q_{12} \qquad (36)$$

$$(x^2 + y^2)^2 = |\lambda|^4 + 2|\lambda|^2 q + 2\,\mathrm{Re}\,\lambda^2 \omega q_{12} + (1 - |\omega|^2)(q^2 - |q_{12}|^2)\,. \qquad (37)$$

It follows from (36) that q is real as long as (37) is satisfied for some q_{12}. The condition $q \geq |q_{12}|$ turns (37) into the inequality

$$(x^2 + y^2)^2 \geq |\lambda|^4 + 2|\lambda|^2 q + 2\,\mathrm{Re}\,\lambda^2 \omega q_{12}. \qquad (38)$$

Now (36) and (38) jointly define regions bounded by a family of octic curves parameterized by ω. The curves are shown for the eigenvalue $\lambda = -1 + 2j$ in Figures 9 – 12 in blue, each figure corresponding to a

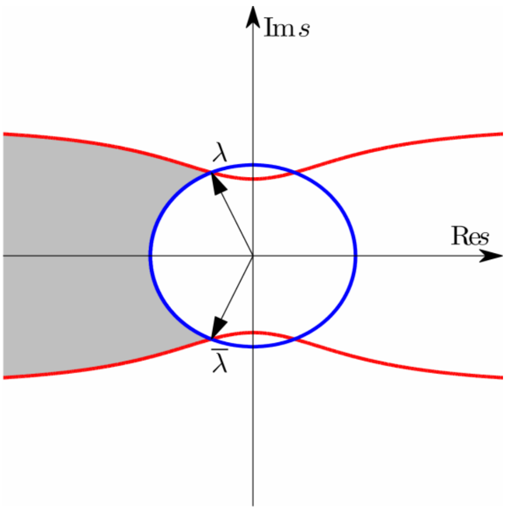

Figure 10.The allowable target region for $\lambda = -1 + 2j$ and for $|\omega| = 0.8$.

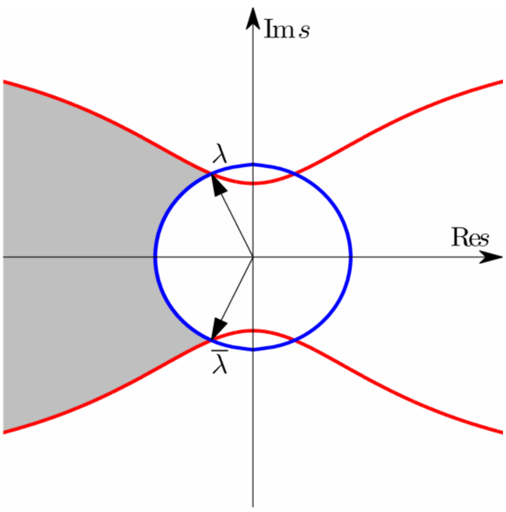

Figure 11.The allowable target region for $\lambda = -1 + 2j$ and for $|\omega| = 0.9$.

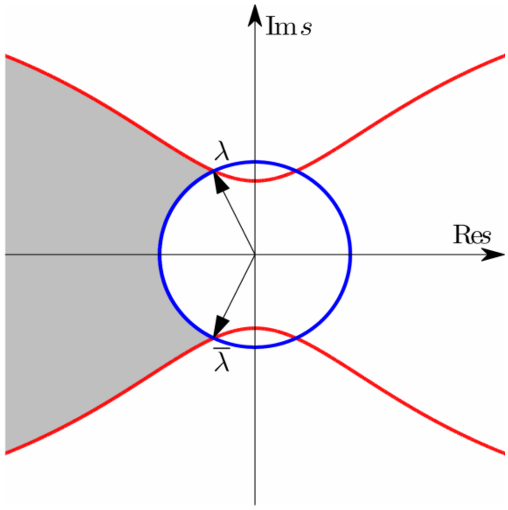

Figure 12.The allowable target region for $\lambda = -1 + 2j$ and for $|\omega| = 0.95$.

particular value of ω. Note that $q = |q_{12}|$ holds along the blue curves and that these curves provide an upper bound for the real part x of μ.

On the other hand, the upper bound for the imaginary part y of μ is evaluated from (36) and (37) for each x. The result is a family of curves parameterized by ω. The curves are shown for the eigenvalue $\lambda = -1 + 2j$ in Figures 9 – 12 in red, each figure corresponding to a particular value of ω. No fixed relationship between q and q_{12} holds along the red lines.

The shaded areas shown in Figures 9 – 12 portray the regions into which $\mu, \overline{\mu}$ can be assigned. Thus the real part of the eigenvalues can be shifted leftward while the imaginary part is bounded from above.

Note that when $\omega \to 0$ the attainable regions shown in Figures 9 – 12 approach the region shown in Figure 5. On the other hand, when $|\omega| \to 1$, we recover the singular case, see Figure 2. It is of interest to note that the maximal assignable imaginary part in Figures 9 – 12 grows progressively with ω. The growth is slow for $\omega < 0.5$ and is fast only when $\omega > 0.9$.

To summarize, one can pick any pair of simple, complex conjugate eigenvalues $\lambda, \overline{\lambda}$ of A and shift it to a desired position $\mu, \overline{\mu}$ within the region defined by λ and ω through equations (23) and (24), while keeping the remaining eigenvalues unchanged. This can be done by solving a simple linear-quadratic regulator problem. The problem has a solution in terms of the left eigenvectors $\overline{w}^T, w^T$ of A associated respectively with $\lambda, \overline{\lambda}$. Taking a matrix

$$Q = \begin{bmatrix} w & \overline{w} \end{bmatrix} Q_2 \begin{bmatrix} \overline{w}^T \\ w_2 \end{bmatrix}$$

in which the entries of Q_2 satisfy (23) and (24) and selecting a matrix R such that (19) holds, namely

$$\begin{bmatrix} \overline{w}^T \\ w^T \end{bmatrix} BR^{-1}B^T \begin{bmatrix} w & \overline{w} \end{bmatrix} = \Omega_2 ,$$

one obtains

$$P = \begin{bmatrix} w & \overline{w} \end{bmatrix} P_2 \begin{bmatrix} \overline{w}^T \\ w^T \end{bmatrix}$$

where P_2 is the solution of equation (22). The feedback gain matrix F that accomplishes this task is given by (4).

The target eigenvalues can in particular be taken real, resulting in a double real eigenvalue μ. This case is addressed by setting $y = \operatorname{Im} \mu = 0$ in the expressions above.

The process can be repeated for each pair of complex conjugate eigenvalues *ad libitum*. We note, however, that their real parts can only be shifted to the left while their imaginary parts are bounded from above, due to the special structure of Q.

6 CONCLUSION

An iterative method has been developed to design linear-quadratic optimal systems with prescribed eigenvalues. The method is well suited to modify a given linear-quadratic design so as to improve the transient response of the closed-loop system. Slow eigenvalues can be made faster and oscillatory eigenvalues can be dampened. A detailed analysis of the shifts possible has been presented, including the case of a complex conjugate pair of eigenvalues. The didactic value of the results can be seen in providing an explicit relationship between the weighting matrices and the closed-loop eigenvalue positions. The method is so simple that it can eventually make its way to control textbooks.

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7 REFERENCES

Alexandridis, A.T. and G.D. Galanos (1987). Optimal pole-placement for linear multi-input controllable systems. *IEEE Trans. CAS*, **34**, 1602-1604.

Amin, M.H. (1985). Optimal pole shifting for continuous multivariable linear systems. *Int. J. Control*, **41**, 701-707.

Anderson, B.D.O. and J.B. Moore (1969). Linear systems optimization with prescribed degree of stability. *IEE Proc. D*, **116**, 2083-2085.

Anderson, B.D.O. and J.B. Moore (1990). *Optimal*

Control: Linear Quadratic Methods. Prentice-Hall, Englewood Cliffs.

Antsaklis, P. and A.N. Michel (1997). *Linear Systems*. McGraw-Hill, New York.

Cigler, J. (2009). *Posunování pólů kvadratickým kritériem*. Diplomová práce, ČVUT – FEL, Praha.

Duplaix, J., G. Enéa, and M. Franceschi (1994). Commande optimale sous contrainte modale. *APII-RAIRO*, **28**, 247-262.

Franceschi, M., G. Enéa, and J. Duplaix (1995). Aggregate pole placement of complex conjugate poles within continuous system optimal control. In: *Proc. IFAC Conf. System Structure and Control*, pp. 410-415. Nantes, France.

Furuta, K. and S.B. Kim (1987). Pole assignment in a specified disk. *IEEE Trans. AC*, **32**, 423-427.

Hench, J.J., C. He, V. Kučera and V. Mehrmann (1998). Dampening controllers via a Riccati equation approach. *IEEE Trans. AC*, **43**, 1280-1284.

Kawasaki, N. and E. Shimemura (1983). Determining quadratic weighting matrices to locate poles in a specific region. *Automatica*, **19**, 557-560.

Kraus, F. J. and V. Kučera (1999). Linear quadratic and pole placement iterative design. In: *Proc. 5th European Control Conf.*, paper F261. Karlsruhe, Germany.

Kučera, V. and F.J. Kraus (1999). Jak kvadratickým kritériem jednotlivé póly přemístit. In: *Proc. 12th Conf. Process Control*, Vol.2, pp. 1-5. Tatranske Matliare, Slovakia.

Seif, M. (1989). Optimal modal controller design by entire eigenstructure assignment. *IEEE Proc. D*, **136**, 341-344.

Solheim, O.A. (1972). Design of optimal control system with prescribed eigenvalues. *Int. J. Control*, **15**, 143-160.

Sugimoto, K. and Y. Yamamoto (1989). On successive pole assignment by linear quadratic optimal feedbacks. *Lin. Alg. Appl.*, **122/123/124**, 697-724.